Chapter 8

Discrete Symmetries

What does it mean by 'a certain law of physics is symmetric under certain transformations'? To be specific, consider the statement 'classical mechanics is symmetric under mirror inversion' which can be defined as follows: take any motion that satisfies the laws of classical mechanics. Then, reflect the motion into a mirror and imagine that the motion in the mirror is *actually happening in front of your eyes*, and check if the motion satisfies the same laws of classical mechanics. If it does, then classical mechanics is said to be symmetric under mirror inversion. Or more precisely, if *all* motions that satisfy the laws of classical mechanics also satisfy them after being reflected into a mirror, then classical mechanics is said to be symmetric under mirror inversion. In general, suppose one applies certain transformation to a motion that follows certain law of physics, if the resulting motion satisfies the same law, and if such is the case for all motion that satisfies the law, then the law of physics is said to be symmetric under the given transformation.

It is important to use exactly the same law of physics after the transformation is applied. For example, I could use my right hand to specify the law physics, say, to state the direction of the force felt by a moving charge inside a magnetic field. Then, I have to use the same hand to see the law is still satisfied by the mirror-inverted motion. If I also mirror-invert my right hand to apply the law, then the law would be trivially satisfied by the transformed motion.

In the context of quantum mechanics, the above criterion for the symmetry of physical law can be stated as follows: for the state vectors $|i\rangle$ and $|f\rangle$ representing certain initial and final states, there exist $|i'\rangle = \mathcal{U}|i\rangle$ and $|f'\rangle = \mathcal{U}|f\rangle$ that represent the corresponding states reflected into a mirror, where $\mathcal{U}$ is an operator in the Hilbert space that corresponds to mirror inversion. Then, if the laws of physics are symmetric under mirror inversion, the transition probability is the same before and after the transformation:

$$|\langle f'|S|i'\rangle|^2 = |\langle f|S|i\rangle|^2 \,. \tag{8.1}$$

Note that the same S operator, not $S' \equiv \mathcal{U}S\mathcal{U}^{-1}$, is used for the transformed states.

In fact, if S' is used instead of S, the transition probability is trivially invariant as long as $\mathcal{U}^\dagger = \mathcal{U}^{-1}$:

$$|\langle f|\underbrace{}_{\mathcal{U}^{-1}\mathcal{U}} S \underbrace{}_{\mathcal{U}^{-1}\mathcal{U}} |i\rangle|^2 = |\langle f'|S'|i'\rangle|^2 \quad \text{(always)}\,. \tag{8.2}$$

In general, a transformation $|\Psi'\rangle = \mathcal{U}|\Psi\rangle$ is called a symmetry transformation if it preserves inner products between any physical states

$$|\langle \Psi_1'|\Psi_2'\rangle|^2 = |\langle \Psi_1|\Psi_2\rangle|^2\,. \tag{8.3}$$

There is a theorem by Wigner which states that the necessary and sufficient condition for a transformation to be a symmetry transformation is that it is unitary and linear or antiunitary and antilinear. The latter turns out to be the case for the time reversal operator, and the term 'antiunitary' and 'antilinear' will be defined when we study the topic, until then we will assume that the symmetry operators we encounter (the mirror inversion and the particle-antiparticle exchange) are linear and unitary. If the symmetry operator $\mathcal{U}$ commutes with the S operator, then

$$S' \overset{\text{def}}{\equiv} \mathcal{U}S\mathcal{U}^{-1} = S \quad \to \quad |\langle f|S|i\rangle|^2 = |\langle f'|S|i'\rangle|^2\,; \tag{8.4}$$

namely, the laws of physics are symmetric under the transformation. In order for a physical law to be symmetric under certain transformation, it is not enough that the transformation is a symmetry transformation. On the other hand, if the symmetry transformation operator commutes with the S operator, then the physics represented by the S operator is symmetric under the given transformation. The above argument applies to any unitary transformation $\mathcal{U}$ including particle-antiparticle exchange as well as mirror inversion.

In the case of time reversal symmetry, we will see that the antilinearity of the time reversal operator $\mathcal{T}$ leads to

$$\mathcal{T}S\mathcal{T}^{-1} = S^\dagger \qquad (\mathcal{T}:\ \text{antilinear}) \tag{8.5}$$

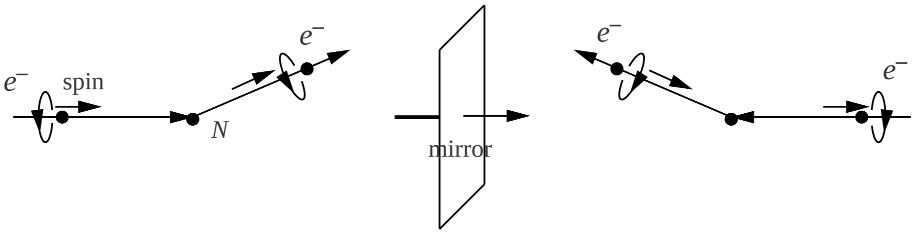
instead of (8.4), which results in exchanging the initial and final states as well as flipping the signs of momenta and spins. We have defined the S operator in the context of the interaction picture, and we will stay in the interaction picture unless otherwise stated.

8.1 Invariance of processes under P and C

Let us start from some concrete examples of symmetries under mirror inversion and the particle-antiparticle exchange.

The parity transformation P is the operation to change the signs of the three space coordinates, which is equivalent to the mirror inversion followed by a rotation by π radian. To be specific, the mirror inversion $z \to -z$ followed by a rotation around the z axis by π flips the sign of all three coordinates. Since rotating a whole motion by π does not affect if the motion will satisfy certain law of physics or not, the parity transformation and the mirror inversion are equivalent. The charge conjugation C changes every particle to its antiparticle and vice versa without changing momentum and spin.

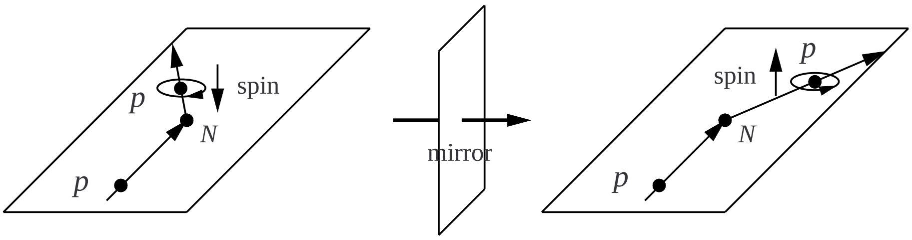
Let's take an example of a Coulomb scattering of an electron by a nucleus N where incoming e^- is polarized right-handed (helicity $+$) and the outgoing e^- is also right-handed.



(8.6)

The nucleus could recoil, but we will focus on the motion of the electron. After the mirror inversion, the electrons in the reflected process are left-handed for both the initial and final state. In fact, by rotating the reflected process by 180 degrees around the vertical axis, one can make it completely overlap with the original process except that the spins are in the opposite direction. If the physics involved is symmetric under parity, the original and reflected processes should occur with the same cross section. This is experimentally confirmed. As far as we know, every process caused by QED and its mirror inversion occur with same probability, and thus we believe that QED is symmetric under parity.

Next, consider an unpolarized proton (p) elastically scattering off an unpolarized target (N).

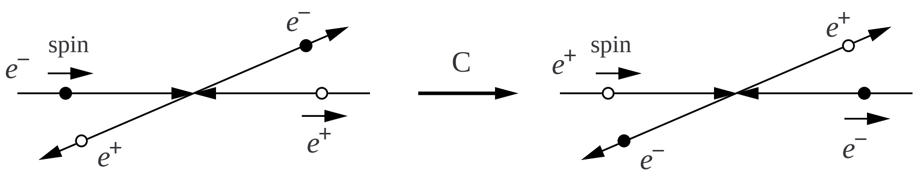


(8.7)

One sees from the figure that, if the proton scattered toward left is transversely polarized in the down direction, then the proton in the reflected process is scattered toward right (for the mirror position as shown) with its spin polarized in the up direction. If the physics responsible for the scattering conserves parity, then the two processes should occur with same cross section, and this is indeed the case experimentally. Thus, such transverse polarization does not indicate that the parity is violated by the underlining physics.

This has a close analogy in the classical mechanics which we know is invariant under parity. Imagine that a tennis ball with no spin hits a basket ball. When the tennis ball hits on the left-hand side of the basket ball (looking from the point of view of the tennis ball) and scatters to the left, the tennis ball would be spinning with its spin angular momentum pointing downward. Similarly, when the tennis ball hits on the right-hand side of the basket ball and scatters to the right, the tennis ball would be spinning with its spin up. Since we know that classical mechanics is symmetric under parity, such transverse 'polarization' could not violate parity.

For the charge conjugation operation C, consider the scattering $e^-e^+ \to e^-e^+$ where the initial state is polarized as shown.



(8.8)

Under charge conjugation, the momenta and spins are kept the same and simply particle and antiparticle are exchanged. If QED is symmetric under C, these two processes should occur at the same rate, which is indeed the case experimentally.

8.2 Parity invariance

Let's first study the parity transformation in some detail. We will first define the parity operator in the Hilbert space which will have a set of arbitrary parity phases, and then will try to pick the phases such that the parity operator commutes with the S operator. If we could do so, then we will see that the law of physics is symmetric under the parity transformation as defined above.

8.2.1 Definition of the parity operator

As we have seen in (1.42), the space inversion P belongs to the Lorentz group and it is explicitly written in the space-time four-dimension as

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{8.9}$$

In this section, we define the corresponding parity operator in the Hilbert space which denote as $\mathcal{P}$.

We first note that once we specify the action of a linear operator on each basis state of the Hilbert space, then the linear operator is uniquely defined. The basis of the Hilbert space may be taken as

$$|\{n_i, \vec{p}_i, \sigma_i\}\rangle \overset{\text{def}}{\equiv} a^\dagger_{n_1\vec{p}_1\sigma_1} \cdots a^\dagger_{n_k\vec{p}_k\sigma_k}|0\rangle \tag{8.10}$$

which is a k-particle state where i-th particle is specified by the particle type n_i, its momentum $\vec{p}_i$, and the spin component along some axis σ_i. An antiparticle of type n will be indicated by $\bar{n}$ which is included in n_i; namely, antiparticle is treated as just another particle type and there will be no 'b' operators. Bound states are included as separate particles. The axis of spin quantization could be taken to be different for different particle type and momentum, but we will take it as the z direction for all particle types and momenta for simplicity.

The momentum index is the eigenvalue of the total momentum operator $\vec{P}$:

$$\vec{P}|n, \vec{p}, \sigma\rangle = \vec{p}|n, \vec{p}, \sigma\rangle \, , \tag{8.11}$$

where $\vec{P}$ is defined by the standard construction (4.116) and (4.117). The energy of a one-particle state is given by

$$H|n, \vec{p}, \sigma\rangle = p^0|n, \vec{p}, \sigma\rangle \, , \quad p^0 \overset{\text{def}}{\equiv} \sqrt{\vec{p}^2 + m^2} \, , \tag{8.12}$$

where H is the total Hamiltonian again constructed by the standard procedure from the Lagrangian and m is the physical mass of the particle. For a multiple-particle state, the action of $P^\mu \equiv (H, \vec{P})$ is given by

$$P^\mu|\{n_i, \vec{p}_i, \sigma_i\}\rangle = \Big(\sum_i p_i^\mu\Big)|\{n_i, \vec{p}_i, \sigma_i\}\rangle \, , \quad p_i^0 \overset{\text{def}}{\equiv} \sqrt{\vec{p}_i^2 + m_i^2} \, . \tag{8.13}$$

In the presence of interaction, the basis states are also eigenstates of the free field part of the Hamiltonian H_0, and we assume that the Hamiltonian is renormalized,

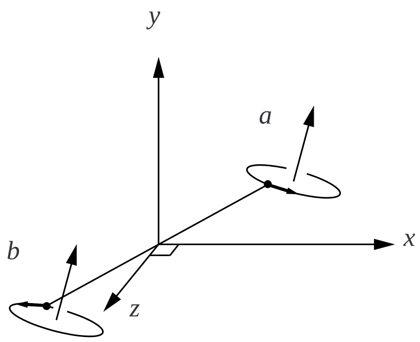


Figure 8.1: Transformation of classical angular momentum under the parity inversion. Angular momentum does not change under parity.

namely, the masses that appear in the free field part are the physical masses. Implicit in the definition of the basis state (8.10) is that if there are N particles with exactly the same type, momentum, and spin, then there is a factor $1/\sqrt{N!}$ to ensure the normalization

$$\langle\{n_i,\vec{p}_i,\sigma_i\}|\{n_i,\vec{p}_i,\sigma_i\}\rangle = 1\,. \tag{8.14}$$

The complete set of the basis states is then formed by taking all possible combinations of the particle types, momenta, spins, and number of particles. Once the action of the parity operator $\mathcal{P}$ on each creation operator is specified, its action on any basis state is defined, and thus the operator is uniquely defined.

Under the space inversion, the 3-momentum $\vec{p}$ changes its sign, the particle type n stays the same, and the spin component σ is unchanged. The last point is seen by the classical example in Figure 8.1 where the angular momentum of a particle is invariant under the space inversion. Thus, the parity operator $\mathcal{P}$ should transform a creation operator of particle type n, momentum $\vec{p}$, and spin σ to that for $(n,-\vec{p},\sigma)$ up to a phase that in general depends on $(n,\vec{p},\sigma)$:

$$\boxed{\mathcal{P}a^\dagger_{n\vec{p}\sigma}\mathcal{P}^{-1} = \eta_{n\vec{p}\sigma}\,a^\dagger_{n\,-\vec{p}\,\sigma}\,. \quad (|\eta_{n\vec{p}\sigma}|=1)} \tag{8.15}$$

Noting that the vacuum $|0\rangle$ should transform to itself under parity up to a phase

$$\mathcal{P}|0\rangle = \eta_{\rm VAC}\,|0\rangle\,, \tag{8.16}$$

the basis state $|\{n_i,\vec{p}_i,\sigma_i\}\rangle$ of (8.10) transforms as

$$\mathcal{P}|\{n_i,\vec{p}_i,\sigma_i\}\rangle \;=\; \mathcal{P}a^\dagger_{n_1\vec{p}_1\sigma_1}\underbrace{\cdots}_{\mathcal{P}^{-1}\mathcal{P}} \qquad \underbrace{\cdots}_{\mathcal{P}^{-1}\mathcal{P}}\, a^\dagger_{n_k\vec{p}_k\sigma_k}\underbrace{\;}_{\mathcal{P}^{-1}\mathcal{P}}|0\rangle$$

$$\begin{aligned} &= \underbrace{\eta_{n_1\vec{p}_1\sigma_1}\cdots\eta_{n_k\vec{p}_k\sigma_k}}_{\equiv\,\eta_{\{n_i\vec{p}_i\sigma_i\}}}\eta_{\rm VAC}a^\dagger_{n_1\,-\vec{p}_1\,\sigma_1}\cdots a^\dagger_{n_k\,-\vec{p}_k\,\sigma_k}|0\rangle \\ &= \eta_{\{n_i\vec{p}_i\sigma_i\}}\eta_{\rm VAC}\,|\{n_i\,-\vec{p}_i\,\sigma_i\}\rangle\,, \end{aligned} \tag{8.17}$$

where we have defined the combined parity phase

$$\eta_{\{n_i\vec{p}_i\sigma_i\}} \equiv \eta_{n_1\vec{p}_1\sigma_1}\cdots\eta_{n_k\vec{p}_k\sigma_k}\,. \tag{8.18}$$

This gives a large matrix representing the transformation of the basis states and uniquely defines the parity operator $\mathcal{P}$ in the Hilbert space.

The vacuum parity phase $\eta_{\rm VAC}$ is just an overall phase of $\mathcal{P}$ and can be absorbed into the definition of $\mathcal{P}$. Namely, defining

$$\mathcal{P}' \stackrel{\rm def}{\equiv} \eta^*_{\rm VAC}\mathcal{P}\,, \tag{8.19}$$

the (8.16) becomes

$$\mathcal{P}'|0\rangle = |0\rangle\,. \tag{8.20}$$

We redefined the parity operator leaving the vacuum the same, but this amounts to choosing the *intrinsic parity* of the vacuum to be +1. Note that this redefinition of $\mathcal{P}$ does not change the parity phase $\eta_{n,\vec{p},\sigma}$ in (8.15) for the same creation operators $a^\dagger_{n\vec{p}\sigma}$ and $a^\dagger_{n\,-\vec{p}\,\sigma}$. Hereafter, we assume that the parity of the vacuum is taken to be +1; thus

$$\begin{aligned} \mathcal{P}|0\rangle &= |0\rangle \\ \mathcal{P}|\{n_i,\vec{p}_i,\sigma_i\}\rangle &= \eta_{\{n_i,\vec{p}_i,\sigma_i\}}|\{n_i,-\vec{p}_i,\sigma_i\}\rangle \end{aligned} \tag{8.21}$$

For each basis state there is only one basis it transforms to under $\mathcal{P}$. It means that any given row of the large matrix that represents $\mathcal{P}$ has only one non-zero element which is a phase factor, and no two rows have the same columns non-zero. Thus, the vectors represented by the rows are orthonormal; namely, the matrix is unitary:

$$\mathcal{P}^\dagger\mathcal{P} = \mathcal{P}\mathcal{P}^\dagger = 1 \quad\to\quad \mathcal{P}^{-1} = \mathcal{P}^\dagger\,. \tag{8.22}$$

We have seen that for the set of creation operators $\{a^\dagger_{n_i,\vec{p}_i,\sigma_i}\}$ that corresponds to all possible particle types, momenta, and spins, one can pick an arbitrary set of phases $\{\eta_{n_i,\vec{p}_i,\sigma_i}\}$ to construct an unitary operator $\mathcal{P}$ that represents space inversion in the Hilbert space. Any phases $\{\eta_{n_i,\vec{p}_i,\sigma_i}\}$ will do at this point.

8.2.2 Conservation of parity quantum number

As we have seen earlier, when the parity operator $\mathcal{P}$ commutes with the S operator, then transition probabilities stay the same when the initial and final states are parity-inverted:

$$\mathcal{P}S\mathcal{P}^{-1} = S \quad\to\quad \langle f|\underbrace{\ }_{\mathcal{P}^\dagger\mathcal{P}}S\underbrace{\ }_{\mathcal{P}^\dagger\mathcal{P}}|i\rangle = \langle Pf|S|Pi\rangle \tag{8.23}$$

where

$$|Pi\rangle \equiv \mathcal{P}|i\rangle\,, \quad \text{and} \quad |Pf\rangle \equiv \mathcal{P}|f\rangle \tag{8.24}$$

represents the states with their momenta flipped while particle types and spins are kept the same.

Now suppose the initial and final states happen to be eigenstates of $\mathcal{P}$ with eigen values η_i and η_f respectively:

$$\mathcal{P}|i\rangle = \eta_i|i\rangle\,, \qquad \mathcal{P}|f\rangle = \eta_f|f\rangle\,, \tag{8.25}$$

where η_i and η_f are phase factors. Furthermore, assume that the physics is symmetric under parity - namely, the parity operator $\mathcal{P}$ commutes with the S operator. Then, we have

$$S_{f\,i} \equiv \langle f|\underbrace{}_{\mathcal{P}^\dagger\mathcal{P}}\, S\, \underbrace{}_{\mathcal{P}^\dagger\mathcal{P}}|i\rangle = \eta_f^*\eta_i\,\langle f|S|i\rangle = \eta_f^*\eta_i S_{f\,i}\,. \tag{8.26}$$

This means that the transition amplitude vanishes unless $\eta_f^*\eta_i = 1$, or multiplying η_f on both sides

$$\eta_f = \eta_i\,; \qquad (\text{otherwise } S_{f\,i} = 0) \tag{8.27}$$

namely, the parity eigenvalue is conserved. Thus, in quantum mechanics, one can discuss the symmetry under parity in terms of the conservation of parity eigenvalues. This is a feature that has no direct counter part in the classical mechanics. Both the conservation of the parity eigenvalues as well as the classical interpretation of symmetry under parity given in (8.23) comes down to the commutation of the parity operator with the S operator.

The S operator is given in terms of the interaction Hamiltonian $h(t)$ as [see (5.72)]

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \int dt_1 \cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,, \tag{8.28}$$

where

$$h(t) = \int d^3x \mathcal{H}_{\rm int}(x)\,. \tag{8.29}$$

Now if $\mathcal{P}$ commutes with $A(t)$ and $B(t)$, then it commutes with $T\Big(A(t_1)B(t_2)\Big)$:

$$\begin{aligned} \mathcal{P}\,T\Big(A(t_1)B(t_2)\Big) &= \mathcal{P}\Big[\theta(t_1-t_2)A(t_1)B(t_2) + \theta(t_2-t_1)B(t_2)A(t_1)\Big] \\ &= \Big[\theta(t_1-t_2)A(t_1)B(t_2) + \theta(t_2-t_1)B(t_2)A(t_1)\Big]\mathcal{P} \\ &= T\Big(A(t_1)B(t_2)\Big)\mathcal{P}\,, \end{aligned} \tag{8.30}$$

where we have used the fact that $\theta(t)$ is just a number. Similarly, if $\mathcal{P}$ commutes with each element of a time-ordered product of more than two elements, then $\mathcal{P}$ commutes with the time-ordered product itself. Thus, if $\mathcal{P}$ commutes with $h(t)$, then it will commute with S. Or in terms of the Hamiltonian density, if

$$\mathcal{P}\mathcal{H}_{\rm int}(x)\mathcal{P}^{-1} = \mathcal{H}_{\rm int}(x') \quad (x' \equiv Px = (x^0, -\vec{x}))\,, \tag{8.31}$$

then $\mathcal{P}$ will commute with $h(t) = \int d^3x \mathcal{H}_{\rm int}(x)$ and thus with S guaranteeing that the law of physics represented by it is symmetric under parity.

We have seen that the parity operator $\mathcal{P}$ that satisfies the natural requirement is not unique; rather, one can pick any arbitrary set of phases $\{\eta_{n_i,\vec{p}_i,\sigma_i}\}$. The name of the game, then, is if one could find a set of phases that makes $\mathcal{P}$ commute with the S operator or satisfy (8.31). If there exists such set of phases, then the law of physics is symmetric under parity. Conversely, if the law of physics is not symmetric under parity, then one cannot choose a set of phases that makes $\mathcal{P}$ commute with S. On the other hand, even if $\mathcal{P}$ does not commute with S for certain set of phases, that does not necessarily mean that the law of physics represented by S is not symmetric under parity.

We will take some sample interactions and choose the parity phases such that $\mathcal{P}$ commutes with the interaction Hamiltonian $h(t)$. We will use the QED interaction which is given up to a real coupling constant by

$$h_{QED} = \int d^3x\, \bar{\psi}\gamma^\mu\psi\, A_\mu\,, \tag{8.32}$$

to find transformation properties of the fermion and the photon fields. For a scalar field, we will take the Yukawa coupling

$$h_Y = \int d^3x\, \bar{\psi}\psi\, \phi\,, \tag{8.33}$$

and require that it commutes with $\mathcal{P}$. As we will see, the parity phases will have to be the same for a given type of particle, namely, independent of momentum and spin. This phase factor is called the *intrinsic parity* of the particle. Then, the intrinsic parity of a paricle is the eigenvalue of $\mathcal{P}$ for a single particle at rest as can be seen from (8.21):

$$\mathcal{P}|n,\vec{0},\sigma\rangle = \eta_{n,\vec{0},\sigma}|n,\vec{0},\sigma\rangle\,. \tag{8.34}$$

The intrinsic parities will be chosen such that $\mathcal{P}$ will commute with S representing the given interaction. It should be noted that we are taking the above interactions just as examples, and the choice of intrinsic parity for a given field may differ for different interactions. There will be cases where, with certain set of intrinsic parities, $\mathcal{P}$ commutes with interaction A and does not commute with interaction B, while with a different set of intrinsic parities $\mathcal{P}$ will commute with B and does not commute with A. The choice, then, will be in principle artificial.

8.2.3 Transformation of fields under $\mathcal{P}$

Our starting point is the momentum expansions of fields in terms of creation and annihilation operators:

$$\begin{aligned}
\phi(x) &= \sum_{\vec{p}} \Big(a_{n\vec{p}}\, e_{\vec{p}}(x) + a^{\dagger}_{\bar{n}\vec{p}}\, e^{*}_{\vec{p}}(x) \Big) && \text{(spin 0)} \\
\psi(x) &= \sum_{\vec{p},\sigma} \Big(a_{n\vec{p}\sigma} f_{\vec{p}\sigma}(x) + a^{\dagger}_{\bar{n}\vec{p}\sigma} g_{\vec{p}\sigma}(x) \Big) && \text{(spin 1/2)} \\
A^{\mu}(x) &= \sum_{\vec{p},\sigma} \Big(a_{n\vec{p}\sigma} \epsilon^{\mu}_{\vec{p}\sigma} e_{\vec{p}}(x) + a^{\dagger}_{\bar{n}\vec{p}\sigma} \epsilon^{\mu *}_{\vec{p}\sigma} e^{*}_{\vec{p}}(x) \Big) && \text{(spin 1)}
\end{aligned} \tag{8.35}$$

with

$$e_{\vec{p}}(x) \equiv \frac{e^{-ip\cdot x}}{2p^0 V}\,, \quad f_{\vec{p}\sigma} \equiv u_{\vec{p}\sigma} e_{\vec{p}}(x)\,, \quad g_{\vec{p}\sigma} \equiv v_{\vec{p}\sigma} e^{*}_{\vec{p}}(x)\,. \tag{8.36}$$

These are the same as what we have seen before except that the annihilation operators of antiparticles are written as $a_{\bar{n}}$ rather than b. The derivation of the momentum expansions applied to any spin quantization axis which is taken to be the z axis. Still, you may be wondering if the creation operators used in the momentum expansion of fields above are really the same operators as what we have constructed the basis states of the Hilbert space with. They may indeed differ by phases. The relations from which essentially all fundamental features of quantum field theory arise are the commutation and anticommutation relations among creation and annihilation operators

$$[a_{n\vec{p}\sigma}, a^{\dagger}_{n'\vec{p}'\sigma'}] = \delta_{n,n'}\delta_{\vec{p},\vec{p}'}\delta_{\sigma,\sigma'} \qquad \text{(for bosons)} \tag{8.37}$$

$$\{a_{n\vec{p}\sigma}, a^{\dagger}_{n'\vec{p}'\sigma'}\} = \delta_{n,n'}\delta_{\vec{p},\vec{p}'}\delta_{\sigma,\sigma'} \qquad \text{(for fermions)} \tag{8.38}$$

and these relations are invariant under redefinition of the operators by

$$a_{n\vec{p}\sigma} \to e^{i\phi_{n\vec{p}\sigma}} a_{n\vec{p}\sigma}\,. \tag{8.39}$$

where the phase is in general a function of $n, \vec{p}, \sigma$. Thus, we can assume that the creation operators used for the basis states are the same as those appearing in the momentum expansion of fields.

The transformation of creation and annihilation operators under parity is

$$\mathcal{P} a^{\dagger}_{n\vec{p}\sigma} \mathcal{P}^{\dagger} = \eta_{n\vec{p}\sigma}\, a^{\dagger}_{n\,-\vec{p}\,\sigma} \quad \xrightarrow{\text{take } \dagger} \quad \mathcal{P} a_{n\vec{p}\sigma} \mathcal{P}^{\dagger} = \eta^{*}_{n\vec{p}\sigma}\, a_{n\,-\vec{p}\,\sigma}\,. \tag{8.40}$$

Let's first apply the parity operator to a spin-0 field $\phi(x)$

$$\phi(x) = \sum_{\vec{p}} \Big(a_{s\vec{p}}\, e_{\vec{p}}(x) + a^{\dagger}_{\bar{s}\vec{p}}\, e^{*}_{\vec{p}}(x) \Big)\,. \tag{8.41}$$

where we have set $n = s$ to indicate that it is a scalar field. If it is a neutral scalar field, we have $\bar{s} = s$. Using (8.40), we have

$$\begin{aligned}
\mathcal{P}\phi(x)\mathcal{P}^\dagger &= \sum_{\vec{p}} \Big(\underbrace{\mathcal{P}a_{s\vec{p}}\mathcal{P}^\dagger}_{\eta^*_{s\vec{p}}\, a_{s\,-\vec{p}}} \; e_{\vec{p}}(x) + \underbrace{\mathcal{P}a^\dagger_{\bar{s}\vec{p}}\mathcal{P}^\dagger}_{\eta_{\bar{s}\vec{p}}\, a^\dagger_{\bar{s}\,-\vec{p}}} \; e^*_{\vec{p}}(x)\Big) \\
&= \sum_{\vec{p}} \frac{1}{\sqrt{2p^0V}} \Big(\eta^*_{s\vec{p}}\, a_{s\,-\vec{p}}\, e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})} + \eta_{\bar{s}\vec{p}}\, a^\dagger_{\bar{s}\,-\vec{p}}\, e^{i(p^0x^0 - \vec{p}\cdot\vec{x})}\Big) \\
\Big(x' = (x^0, -\vec{x})\Big) \quad &= \sum_{\vec{p}} \frac{1}{\sqrt{2p^0V}} \Big(\eta^*_{s\vec{p}}\, a_{s\,-\vec{p}}\, e^{-i(p^0x'^0 + \vec{p}\cdot\vec{x}\,')} + \eta_{\bar{s}\vec{p}}\, a^\dagger_{\bar{s}\,-\vec{p}}\, e^{i(p^0x'^0 + \vec{p}\cdot\vec{x}\,')}\Big) \\
(\text{relabel } \vec{p} \to -\vec{p}) \quad &= \sum_{\vec{p}} \frac{1}{\sqrt{2p^0V}} \Big(\eta^*_{s\,-\vec{p}}\, a_{s\vec{p}}\, e^{-ip\cdot x'} + \eta_{\bar{s}\,-\vec{p}}\, a^\dagger_{\bar{s}\vec{p}}\, e^{ip\cdot x'}\Big) \\
&= \sum_{\vec{p}} \Big(\eta^*_{s\,-\vec{p}}\, a_{s\vec{p}}\, e_{\vec{p}}(x') + \eta_{\bar{s}\,-\vec{p}}\, a^\dagger_{\bar{s}\vec{p}}\, e^*_{\vec{p}}(x')\Big)\,. \qquad (8.42)
\end{aligned}$$

We will leave it as it is and move on to the fermion case.

The momentum expansion of a fermion field is

$$\psi(x) = \sum_{\vec{p},\sigma} \Big(a_{f\vec{p}\sigma}\, f_{\vec{p}\sigma}(x) + a^\dagger_{\bar{f}\vec{p}\sigma}\, g_{\vec{p}\sigma}(x)\Big)\,. \qquad (8.43)$$

where we set $n = f$ to indicate that it is a fermion field. Applying $\mathcal{P}$ to this and using (8.40), we get

$$\begin{aligned}
\mathcal{P}\psi(x)\mathcal{P}^\dagger &= \sum_{\vec{p},\sigma} \frac{1}{\sqrt{2p^0V}} \Big(\underbrace{\mathcal{P}a_{f\vec{p}\sigma}\mathcal{P}^\dagger}_{\eta^*_{f\vec{p}\sigma}\, a_{f-\vec{p}\sigma}} \; u_{\vec{p}\sigma}\, e^{-ip\cdot x} + \underbrace{\mathcal{P}a^\dagger_{\bar{f}\vec{p}\sigma}\mathcal{P}^\dagger}_{\eta_{\bar{f}\vec{p}\sigma}\, a^\dagger_{\bar{f}-\vec{p}\sigma}} \; v_{\vec{p}\sigma}\, e^{ip\cdot x}\Big) \\
&\begin{pmatrix} x' = (x^0, -\vec{x}) \\ \text{relabel } \vec{p} \to -\vec{p} \end{pmatrix} \\
&= \sum_{\vec{p},\sigma} \frac{1}{\sqrt{2p^0V}} \Big(\eta^*_{f-\vec{p}\sigma}\, a_{f\vec{p}\sigma}\, u_{-\vec{p}\sigma}\; e^{-ip\cdot x'} + \eta_{\bar{f}-\vec{p}\sigma}\, a^\dagger_{\bar{f}\vec{p}\sigma}\, v_{-\vec{p}\sigma}\; e^{ip\cdot x'}\Big)\,. \qquad (8.44)
\end{aligned}$$

At this point, we would like to express $u_{-\vec{p}\vec{s}}$ and $v_{-\vec{p}\vec{s}}$ in terms of $u_{\vec{p}\vec{s}}$ and $v_{\vec{p}\vec{s}}$ so that the transformed field can be written in a form closely related to the original field. We recall that $u_{\vec{p}\sigma}$ and $v_{\vec{p}\sigma}$ were constructed by boosting the at-rest solutions $u_{\vec{0}\sigma}$ and $v_{\vec{0}\sigma}$ by the boost matrix $\exp(\vec{\xi}\cdot\vec{\alpha}/2)$:

$$u_{\vec{p}\sigma} = e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} u_{\vec{0}\sigma}\,, \quad v_{\vec{p}\sigma} = e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} v_{\vec{0}\sigma}\,, \qquad (8.45)$$

similarly

$$u_{-\vec{p}\sigma} = e^{-\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} u_{\vec{0}\sigma}\,, \quad v_{-\vec{p}\sigma} = e^{-\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} v_{\vec{0}\sigma}\,, \qquad (8.46)$$

where

$$\vec{\xi} = \xi \hat{p} \quad \text{with} \quad \tanh \xi = \frac{|\vec{p}|}{p^0} \,. \tag{8.47}$$

Since $\alpha_i = \gamma^0 \gamma^i$, γ^0 anticommutes with α_i and so with $\vec{\xi} \cdot \vec{\alpha}$. Then, using the definition $\exp(A) \equiv \sum_k A^k/k!$,

$$\boxed{\gamma^0}\overrightarrow{(\vec{\xi} \cdot \vec{\alpha})^n} = (-\vec{\xi} \cdot \vec{\alpha})^n \gamma^0 \quad \to \quad \gamma^0 e^{\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} = e^{-\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} \gamma^0 \,. \tag{8.48}$$

On the other hand, since $\gamma^0 = \not{p}/m$ ($p^\mu = (m, \vec{0})$) is the energy sign operator for the solutions at rest,

$$\gamma^0 u_{\vec{0}\,\sigma} = u_{\vec{0}\,\sigma} \,, \quad \gamma^0 v_{\vec{0}\,\sigma} = -v_{\vec{0}\,\sigma} \,. \tag{8.49}$$

Then, left-multiplying (8.45) by γ^0,

$$\gamma^0 u_{\vec{p}\sigma} = \underbrace{\gamma^0 e^{\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}}}_{e^{-\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} \gamma^0} u_{\vec{0}\,\sigma} = e^{-\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} \underbrace{\gamma^0 u_{\vec{0}\,\sigma}}_{u_{\vec{0}\,\sigma}} = u_{-\vec{p}\sigma} \,, \tag{8.50}$$

$$\gamma^0 v_{\vec{p}\sigma} = \underbrace{\gamma^0 e^{\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}}}_{e^{-\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} \gamma^0} v_{\vec{0}\,\sigma} = e^{-\frac{1}{2}\vec{\xi} \cdot \vec{\alpha}} \underbrace{\gamma^0 v_{\vec{0}\,\sigma}}_{-v_{\vec{0}\,\sigma}} = -v_{-\vec{p}\sigma} \,. \tag{8.51}$$

Note that these relations are independent of the representation of gamma matrices. Using them in (8.44), we obtain

$$\begin{aligned} \mathcal{P}\psi(x)\mathcal{P}^\dagger &= \sum_{\vec{p},\sigma} \frac{1}{\sqrt{2p^0 V}} \left(\eta^*_{f-\vec{p}\sigma}\, a_{f\vec{p}\sigma}\, \gamma^0 u_{\vec{p}\sigma}\, e^{-ip\cdot x'} - \eta_{\bar{f}-\vec{p}\sigma}\, a^\dagger_{\bar{f}\vec{p}\sigma}\, \gamma^0 v_{\vec{p}\sigma}\, e^{ip\cdot x'} \right) \\ &= \gamma^0 \sum_{\vec{p},\sigma} \left(\eta^*_{f-\vec{p}\sigma}\, a_{f\vec{p}\sigma}\, f_{\vec{p}\sigma}(x') - \eta_{\bar{f}-\vec{p}\sigma}\, a^\dagger_{\bar{f}\vec{p}\sigma}\, g_{\vec{p}\sigma}(x') \right) . \end{aligned} \tag{8.52}$$

We proceed similarly for the charged vector field

$$A^\mu(x) = \sum_{\vec{p},\sigma} \left(a_{v\,\vec{p}\sigma} \epsilon^\mu_{\vec{p}\sigma} e_{\vec{p}}(x) + a^\dagger_{\bar{v}\,\vec{p}\sigma} \epsilon^{\mu *}_{\vec{p}\sigma} e^*_{\vec{p}}(x) \right) , \tag{8.53}$$

where we set $n = v$ to indicate that it is a vector field and $\bar{v} = v$ for a neutral spin-1 field. This time, we will need to relate $\epsilon^\mu_{-\vec{p}\sigma}$ to $\epsilon^\mu_{\vec{p}\sigma}$. We recall that $\epsilon^\mu_{\vec{p}\sigma}$ is obtained by boosting the polarization vector at rest $\epsilon^\mu_{\vec{0}\,\sigma} = (0, \hat{\epsilon}_{\vec{p}\sigma})$:

$$\epsilon_{\vec{p}\sigma} \equiv \Lambda_{\vec{p}}\, \epsilon_{\vec{0}\,\sigma} \,, \tag{8.54}$$

where $\Lambda_{\vec{p}}$ is the boost that transforms a mass at rest to $\vec{p}$ and implicitly depends on m. The matrix $\Lambda_{\vec{p}}$ can be explicitly written as (see p33)

$$\Lambda_{\vec{p}} = e^{\vec{\xi} \cdot \vec{K}} \,. \tag{8.55}$$

Using the explicit form (8.55) of the boost $\Lambda_{\vec{p}}$ and the matrix relation $(e^A)^{-1} = e^{-A}$, we see that

$$\Lambda_{-\vec{p}} = e^{-\vec{\xi}\cdot\vec{K}} = \Lambda_{\vec{p}}^{-1}\,. \tag{8.56}$$

On the other hand, left-multiplying G to the defining property $\Lambda_{\vec{p}}^T G \Lambda_{\vec{p}} = G$ (1.36), we get

$$G\Lambda_{\vec{p}}^T G \Lambda_{\vec{p}} = 1 \quad \to \quad \Lambda_{\vec{p}}^{-1} = G\Lambda_{\vec{p}}^T G\,. \tag{8.57}$$

Since K_i are symmetric matrices, so is $\vec{\xi}\cdot\vec{K}$, and thus $\exp(\vec{\xi}\cdot\vec{K})$ is also symmetric; namely, a pure boost is symmetric: $\Lambda_{\vec{p}}^T = \Lambda_{\vec{p}}$. Since $P = G$ numerically, we then have

$$P\Lambda_{\vec{p}}P = G\Lambda_{\vec{p}}^T G \overset{\text{by (8.57)}}{=} \Lambda_{\vec{p}}^{-1} \overset{\text{by (8.56)}}{=} \Lambda_{-\vec{p}}\,. \tag{8.58}$$

Then, using $P\Lambda_{\vec{p}}P = \Lambda_{-\vec{p}}$,

$$\epsilon_{-\vec{p}\sigma} \equiv \Lambda_{-\vec{p}}\,\epsilon_{\vec{0}\,\sigma} = P\Lambda_{\vec{p}}\underbrace{P\epsilon_{\vec{0}\,\sigma}}_{-\epsilon_{\vec{0}\,\sigma}\quad \leftarrow\, \epsilon_{\vec{0}\,\sigma} = (0,\hat{\epsilon}_{\vec{p}\sigma})} = -P\Lambda_{\vec{p}}\,\epsilon_{\vec{0}\,\sigma} = -P\epsilon_{\vec{p}\sigma}\,, \tag{8.59}$$

which may be written as (using the 'accidental' relation $(P\epsilon)^\mu = \epsilon_\mu$)

$$\epsilon^\mu_{-\vec{p}\sigma} = -\epsilon_{\vec{p}\sigma\,\mu}\,. \tag{8.60}$$

Now, applying $\mathcal{P}$ to (8.53) and relabeling $\vec{p} \to -\vec{p}$,

$$\begin{aligned}
\mathcal{P}A^\mu(x)\mathcal{P}^\dagger &= \sum_{\vec{p},\sigma}\Big(\eta^*_{v\,\vec{p}\sigma}a_{v\,-\vec{p}\sigma}\epsilon^\mu_{\vec{p}\sigma}e_{\vec{p}}(x) + \eta_{\bar{v}\,\vec{p}\sigma}a^\dagger_{\bar{v}\,-\vec{p}\sigma}\epsilon^{\mu*}_{\vec{p}\sigma}e^*_{\vec{p}}(x)\Big) \\
&= \sum_{\vec{p},\sigma}\Big(\eta^*_{v\,-\vec{p}\sigma}a_{v\,\vec{p}\sigma}\underbrace{\epsilon^\mu_{-\vec{p}\sigma}}_{-\epsilon_{\vec{p}\sigma\,\mu}}e_{\vec{p}}(x') + \eta_{\bar{v}\,-\vec{p}\sigma}a^\dagger_{\bar{v}\,\vec{p}\sigma}\underbrace{\epsilon^{\mu*}_{-\vec{p}\sigma}}_{-\epsilon^*_{\vec{p}\sigma\,\mu}}e^*_{\vec{p}}(x')\Big) \\
&= -\sum_{\vec{p},\sigma}\Big(\eta^*_{v\,-\vec{p}\sigma}a_{v\,\vec{p}\sigma}\epsilon_{\vec{p}\sigma\,\mu}e_{\vec{p}}(x') + \eta_{\bar{v}\,-\vec{p}\sigma}a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^*_{\vec{p}\sigma\,\mu}e^*_{\vec{p}}(x')\Big)\,.
\end{aligned} \tag{8.61}$$

We will now choose the parity phases such that the Yukawa interaction Hamiltonian $h_Y(t)$ commute with $\mathcal{P}$, or in terms of Hamiltonian density

$$\mathcal{P}\,\bar{\psi}(x)\psi(x)\,\phi(x)\,\mathcal{P}^\dagger = \bar{\psi}(x')\psi(x')\,\phi(x') \quad (x' = Px = (x^0, -\vec{x}))\,. \tag{8.62}$$

We assume that the spin-0 field $\phi(x)$ is hermitian. This assumption, however, is not essential; otherwise we just have to add the hermitian conjugate term to the interaction Hamiltonian, and major results below will stay the same. Taking the conjugate of (8.52)

$$\mathcal{P}\bar{\psi}(x)\mathcal{P}^\dagger = \sum_{\vec{p},\sigma}\Big(\eta_{f-\vec{p}\sigma}\,a^\dagger_{f\vec{p}\sigma}\,\bar{f}_{\vec{p}\sigma}(x') - \eta^*_{\bar{f}-\vec{p}\sigma}\,a_{\bar{f}\vec{p}\sigma}\,\bar{g}_{\vec{p}\sigma}(x')\Big)\gamma^0\,. \tag{8.63}$$

Using this and (8.42), we have

$$\begin{aligned}
\mathcal{P}\underbrace{\bar{\psi}(x)}_{\mathcal{P}^\dagger\mathcal{P}}\underbrace{\psi(x)}_{\mathcal{P}^\dagger\mathcal{P}}\phi(x)\mathcal{P}^\dagger &= \sum_{\vec{p},\sigma}\Big(\underbrace{\eta_{f-\vec{p}\sigma}\,a^\dagger_{f\vec{p}\sigma}\,\bar{f}_{\vec{p}\sigma}(x')}_{(a)} - \eta^*_{\bar{f}-\vec{p}\sigma}\,a_{\bar{f}\vec{p}\sigma}\,\bar{g}_{\vec{p}\sigma}(x')\Big)\cancel{\gamma^0} \\
&\times \cancel{\gamma^0}\sum_{\vec{q},\rho}\Big(\underbrace{\eta^*_{f-\vec{q}\rho}\,a_{f\vec{q}\rho}\,f_{\vec{q}\rho}(x')}_{(b)}\underbrace{-\eta_{\bar{f}-\vec{q}\rho}\,a^\dagger_{\bar{f}\vec{q}\rho}\,g_{\vec{q}\rho}(x')}_{(c)}\Big) \\
&\times \sum_{\vec{k}}\Big(\underbrace{\eta^*_{s\,-\vec{k}}\,a_{s\vec{k}}\,e_{\vec{k}}(x')}_{(d)} + \underbrace{\eta_{\bar{s}\,-\vec{k}}\,a^\dagger_{\bar{s}\vec{k}}\,e^*_{\vec{k}}(x')}_{(e)}\Big)\,, \qquad (8.64)
\end{aligned}$$

which we want to be equal to $\mathcal{H}_{\rm int}(x') = \bar{\psi}(x')\psi(x')\phi(x')$:

$$\begin{aligned}
\bar{\psi}(x')\psi(x')\phi(x') &= \sum_{\vec{p},\sigma}\Big(\underbrace{a^\dagger_{f\vec{p}\sigma}\,\bar{f}_{\vec{p}\sigma}(x')}_{(a)} + a_{\bar{f}\vec{p}\sigma}\,\bar{g}_{\vec{p}\sigma}(x')\Big) \\
&\times \sum_{\vec{q},\rho}\Big(\underbrace{a_{f\vec{q}\rho}\,f_{\vec{q}\rho}(x')}_{(b)} + \underbrace{a^\dagger_{\bar{f}\vec{q}\rho}\,g_{\vec{q}\rho}(x')}_{(c)}\Big) \\
&\times \sum_{\vec{k}}\Big(\underbrace{a_{s\vec{k}}\,e_{\vec{k}}(x')}_{(d)} + \underbrace{a^\dagger_{\bar{s}\vec{k}}\,e^*_{\vec{k}}(x')}_{(e)}\Big)\,, \qquad (8.65)
\end{aligned}$$

Since different products of a's and $a^\dagger$'s are linearly indendent (see the exercise below), each corresponding coefficients in (8.64) and (8.65) should be the same. The terms $(a)\times(c)\times(d)$ gives

$$-\eta_{f-\vec{p}\sigma}\,\eta_{\bar{f}-\vec{q}\rho}\,\eta^*_{s\,-\vec{k}} = 1\,, \qquad (8.66)$$

which should hold for any $(\vec{p},\sigma)$, $(\vec{q},\rho)$, and $\vec{k}$. In particular, for fixed $(\vec{p},\sigma)$ and $(\vec{q},\rho)$, the quantity $\eta_{f-\vec{p}\sigma}\,\eta_{\bar{f}-\vec{q}\rho}$ is a fixed constant; thus, $\eta_{s\,-\vec{k}}$ should be independent of $\vec{k}$. Similarly, $\eta_{f-\vec{p}\sigma}$ should be independent of $(\vec{p},\sigma)$ and $\eta_{\bar{f}-\vec{q}\rho}$ should be independent of $(\vec{q},\rho)$:

$$\eta_{s\,\vec{k}} = \eta_s\,, \qquad \eta_{f\vec{p}\sigma} = \eta_f\,, \qquad \eta_{\bar{f}\vec{p}\sigma} = \eta_{\bar{f}}\,. \qquad (8.67)$$

Thus, the phase relation (8.66) can be written as

$$-\eta_f\eta_{\bar{f}}\eta^*_s = 1\,. \qquad (8.68)$$

Similarly, the terms $(a)\times(c)\times(e)$ gives $-\eta_{f-\vec{p}\sigma}\,\eta_{\bar{f}-\vec{q}\rho}\,\eta_{s\,-\vec{k}} = 1$, which should be valid for any values of $(\vec{p},\sigma)$, $(\vec{q},\rho)$, and $\vec{k}$. In particular, it should hold for fixed values of $(\vec{p},\sigma)$, $(\vec{q},\rho)$ and different values of $\vec{k}$; namely, $\eta_{s\,-\vec{k}}$ should be independent of $\vec{k}$:

$$\eta_{\bar{s}\,\vec{k}} = \eta_{\bar{s}}\,. \qquad (8.69)$$

Thus we can write

$$-\eta_f \eta_{\bar{f}} \eta_{\bar{s}} = 1\,. \tag{8.70}$$

Comparing this with (8.68), we see that

$$\eta_{\bar{s}} = \eta_s^*\,. \tag{8.71}$$

For a hermitian scalar field ($\bar{s} = s$), we then have $\eta_s^* = \eta_s$ which means that the phase factor η_s is real; namely, ± 1. For non-hermitian spin-0 fields, the parity phases are in general not restricted to be real. Now, equating the terms $(a) \times (b) \times (d)$ and using $\eta_{f\vec{p}\sigma} = \eta_f$ and $\eta_{s\vec{k}} = \eta_s$, we obtain,

$$\eta_f \eta_f^* \eta_s^* = 1\,. \tag{8.72}$$

This and (8.68) then gives

$$\eta_{\bar{f}} = -\eta_f^*\,. \tag{8.73}$$

Furthermore, in order for the interaction $h(t)$ to commute with S, (8.72) indicates that the value of η_s should be taken as

$$\eta_s = +1 \qquad (\text{in order for } \mathcal{P} h_Y \mathcal{P}^\dagger = h_Y \text{ to hold})\,. \tag{8.74}$$

Now that we could choose the parity phases such that $h_Y = \bar{\psi}\psi\phi$ commutes with $\mathcal{P}$, we see that the processes caused by this interaction are symmetric under parity.

You may be wondering why we did not use $\eta_f \eta_f^* = 1$ in (8.72) and immediately conclude $\eta_s = 1$. This is because in general the $\psi(x)$ should transform to something proportional to $\psi(x')$ as

$$\mathcal{P}\psi(x)\mathcal{P}^\dagger = (\text{const. matrix})\,\psi(x') \qquad (x' = Px) \tag{8.75}$$

in order for the general interaction to have any hope of commuting with $\mathcal{P}$ since other parts of the interaction made up of other fields cannot fix if $\psi(x)$ does not transform as (8.75). And the condition for (8.75) is $\eta_{\bar{f}} = -\eta_f^*$. Similarly, in order for the spin-0 field $\phi(x)$ to transform to something proportional to $\phi(x')$, one should have $\eta_{\bar{s}} = \eta_s^*$. These conditions amount to requiring that the positive and negative frequency parts of the fields transform the same way under parity. This requirement holds generally for different types of interactions. Using the relations $\eta_{\bar{s}} = \eta_s^*$ and $\eta_{\bar{f}} = -\eta_f^*$ in (8.42) and (8.52) the transformations of spin-0 and spin-1/2 fields are written as

$$\begin{aligned} \mathcal{P}\phi(x)\mathcal{P}^\dagger &= \eta_s^* \phi(x') \\ \mathcal{P}\psi(x)\mathcal{P}^\dagger &= \eta_f^* \gamma^0 \psi(x') \end{aligned} \qquad (x' \equiv Px) \tag{8.76}$$

Thus, the choice of parity phases reduces to picking a phase for each field. If one cannot make the interaction commute with S by doing so, then in general one cannot do so even if one picks differently for particles and antiparticles or for different momenta or spins.

Note that the above form of the parity transformation for a fermion field is exactly the same as what we encountered earlier in (3.120) where we found that $\psi'(x') = \gamma^0\psi(x')$ $(x' = Px)$ also satisfies the Dirac equation in terms of x'. Here, we arrived at the same form starting from the natural requirement for the parity transformation of creation operators and requiring that the interaction $h(t)$ commutes with the parity operator $\mathcal{P}$.

Exercise 8.1 *Linear independence of creation and annihilation operators.*
(*a*) *Suppose $a_{\vec p}$'s are the annihilation operators of a spin-0 particle and $c_{\vec p}$'s and $d_{\vec p}$'s are complex coefficients. Show that*

$$\sum_{\vec p} c_{\vec p} a_{\vec p} = \sum_{\vec p} d_{\vec p} a_{\vec p} \tag{8.77}$$

leads to

$$c_{\vec p} = d_{\vec p}\,. \tag{8.78}$$

[*hint: use the commutation relation of $a_{\vec p}$'s.*]
(*b*) *Suppose the following equation holds:*

$$\sum_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_n} c_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_m} a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_n} a_{\beta_1}\cdots a_{\beta_m} = \sum_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_n} d_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_m} a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_n} a_{\beta_1}\cdots a_{\beta_m} \tag{8.79}$$

where each term is normal-ordered, and each of α_i and β_i represents a set of parameters $(\vec p\,\sigma\,n)$. The sum is over all possible numbers of annihilation and creation operators, and over all possible momenta, spins, and types. Every term in each sum (*the operator part*) *are different: either the number of creation operators are different, that of annihilation operators are different, or any of the indexes are different. If multiple fermion operators are in a given term, assume that they are ordered in a fixed order* (*u-quark comes before d-quark, etc.*) *separately in the creation and annihilation parts. Show that*

$$c_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_m} = d_{\alpha_1\cdots\alpha_n\,\beta_1\cdots\beta_m}\,. \tag{8.80}$$

Next, we will choose parity phases such that the QED interaction Hamiltonian commutes with $\mathcal{P}$. The transformation of the photon field is given by (8.61), and that of the electron field is given by (8.76). Taking the conjugate of the fermion transformation,

$$\mathcal{P}\bar\psi(x)\mathcal{P}^\dagger = \eta_f\bar\psi(x')\gamma^0\,. \tag{8.81}$$

Then the current $j_\mu = \bar{\psi}\gamma_\mu\psi$ transforms as

$$\begin{aligned}\mathcal{P}j_\mu(x)\mathcal{P}^\dagger &= \mathcal{P}\,\bar{\psi}\underbrace{(x)\gamma_\mu}_{\mathcal{P}^\dagger\,\mathcal{P}}\psi(x)\,\mathcal{P}^\dagger = (\eta_f\bar{\psi}(x')\gamma^0)\gamma_\mu(\eta_f^*\gamma^0\psi(x'))\\ &= |\eta_f|^2\bar{\psi}(x')\underbrace{\gamma^0\gamma_\mu\gamma^0}_{\gamma^\mu}\psi(x')\\ &= \bar{\psi}(x')\gamma^\mu\psi(x')\\ &= j^\mu(x')\,,\end{aligned} \tag{8.82}$$

where the index of γ_μ changed from a subscript to a superscript. Namely,

$$\mathcal{P}j_\mu(x)\mathcal{P}^\dagger = j^\mu(x')\,. \tag{8.83}$$

which is consistent with the naive expectation that the space part of the current should change the sign under parity while the time component (i.e. the charge density) should stay the same. This is independent of the parity phase of the fermion field. The transformation of the QED Hamiltonian density $\mathcal{H}_{\rm int} = j_\mu A^\mu$ is then

$$\mathcal{P}j_\mu\underbrace{(x)A^\mu}_{\mathcal{P}^\dagger\mathcal{P}}(x)\mathcal{P}^\dagger = j^\mu(x')(-)\sum_{\vec{p},\sigma}\left(\eta^*_{v\,-\vec{p}\sigma}a_{v\,\vec{p}\sigma}\epsilon_{\vec{p}\sigma\,\mu}e_{\vec{p}}(x') + \eta_{\bar{v}\,-\vec{p}\sigma}a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^*_{\vec{p}\sigma\,\mu}e^*_{\vec{p}}(x')\right), \tag{8.84}$$

which should equal to

$$j^\mu(x')A_\mu(x') = j^\mu(x')\sum_{\vec{p},\sigma}\left(a_{v\,\vec{p}\sigma}\epsilon_{\vec{p}\sigma\,\mu}e_{\vec{p}}(x') + a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^*_{\vec{p}\sigma\,\mu}e^*_{\vec{p}}(x')\right), \tag{8.85}$$

Comparing the coefficients, we first note that the phases $\eta_{v\,\vec{p}\sigma}$ and $\eta_{\bar{v}\,\vec{p}\sigma}$ should not depend on $(\vec{p},\sigma)$ and

$$\eta_{\bar{v}} = \eta_v^*\,, \tag{8.86}$$

which should be satisfied regardless of the parity phase on $j^\mu(x')$ and is a general requirement for a vector field for any form of interaction. Moreover, in order for h_{QED} to commute with $\mathcal{P}$, we should assign

$$\eta_v = -1 \qquad \text{(in order for } \mathcal{P}h_{QED}\mathcal{P}^\dagger = h_{QED} \text{ to hold)}\,. \tag{8.87}$$

Using $\eta_{\bar{v}} = \eta_v^*$ in (8.61), the transformation of a vector field is now written as

$$\mathcal{P}A_\mu(x)\mathcal{P}^\dagger = -\eta_v^* A^\mu(x')\,, \tag{8.88}$$

where the Lorentz index μ changed its position. Again, the parity phase η_v is the eigenvalue of $\mathcal{P}$ for the particle at rest:

$$\mathcal{P}|v,\vec{0},\sigma\rangle = \eta_v|v,\vec{0},\sigma\rangle\,,\quad \mathcal{P}|\bar{v},\vec{0},\sigma\rangle = \eta_v^*|\bar{v},\vec{0},\sigma\rangle\,. \tag{8.89}$$

Thus, the intrinsic parity of photon is -1; or more precisely, if we take the intrinsic parity of photon to be -1, then the QED Hamiltonian commute with $\mathcal{P}$, and since we could choose such phases, processes caused by QED are symmetric under mirror inversion.

Next, let's find out how the derivative of a spin-0 field $\partial_\mu \phi$ transforms under parity. Clearly, the time derivative commutes with parity:

$$\mathcal{P}\frac{d}{dt}\phi(x)\mathcal{P}^\dagger = \frac{d}{dt}\mathcal{P}\phi(x)\mathcal{P} = \eta_n^* \frac{d}{dt}\phi(Px)\,. \tag{8.90}$$

The space derivative requires some care. For simplicity, let's look at the z-derivative only. Using the definition of derivative and noting that Δz is just a real number,

$$\begin{aligned}
\mathcal{P}\frac{d}{dz}\phi(z)\mathcal{P}^\dagger &= \mathcal{P}\lim_{\Delta z\to 0}\frac{\phi(z+\Delta z)-\phi(z)}{\Delta z}\mathcal{P}^\dagger \\
&= \lim_{\Delta z\to 0}\frac{\mathcal{P}\phi(z+\Delta z)\mathcal{P}^\dagger - \mathcal{P}\phi(z)\mathcal{P}^\dagger}{\Delta z} \\
&= \lim_{\Delta z\to 0}\eta_n^*\frac{\phi(-(z+\Delta z))-\phi(-z)}{\Delta z} \\
&= \frac{d}{dz}\eta_n^*\phi(-z) \\
&= \frac{d}{dz}\mathcal{P}\phi(z)\mathcal{P}^\dagger\,.
\end{aligned} \tag{8.91}$$

Together with (8.90), we see that ∂_μ and $\mathcal{P}$ commute. Using

$$x' \equiv Px = (t, -\vec{x}) \quad \to \quad \partial'_\mu = \partial^\mu \quad \left(\partial'_\mu \overset{\text{def}}{\equiv} \frac{\partial}{\partial x'^\mu}\right), \tag{8.92}$$

we can write

$$\mathcal{P}\partial_\mu\phi(x)\mathcal{P}^\dagger = \partial_\mu\Big[\mathcal{P}\phi(x)\mathcal{P}^\dagger\Big] = \eta_n^*\partial_\mu\phi(x') = \eta_n^*\partial'^\mu\phi(x')\,, \tag{8.93}$$

where the Lorentz index μ changed from subscript to superscript. Similarly, the derivative of a vector field transforms as

$$\mathcal{P}\partial_\nu A_\mu(x)\mathcal{P}^\dagger = \partial_\mu \underbrace{\mathcal{P}A_\mu(x)\mathcal{P}^\dagger}_{-\eta_n^* A^\mu(x')} = -\eta_n^*\partial'^\nu A^\mu(x')\,. \tag{8.94}$$

We see that one could write symbolically

$$\mathcal{P}\partial_\mu\mathcal{P}^\dagger = \partial'^\mu\,. \tag{8.95}$$

In summary, the transformation properties of fields under parity are

$$\boxed{\begin{array}{rcll} \mathcal{P}\phi(x)\mathcal{P}^\dagger &=& \eta_n^*\phi(Px) & (\text{spin}-0) \\ \mathcal{P}\psi(x)\mathcal{P}^\dagger &=& \eta_n^*\gamma^0\psi(Px) & (\text{spin}-\frac{1}{2}) \\ \mathcal{P}A_\mu(x)\mathcal{P}^\dagger &=& -\eta_n^* A^\mu(Px) & (\text{spin}-1) \\ \eta_{\bar{n}} &=& \eta_n^* & (\text{spin}-0,1) \\ \eta_{\bar{n}} &=& -\eta_n^* & (\text{spin}-\frac{1}{2}) \end{array}} \tag{8.96}$$

For neutral, namely self-conjugate or hermitian, spin-0 and spin-1 fields, we have $\bar{n} = n$; and thus $\eta_{\bar{n}} = \eta_n^*$ reads

$$\eta_n = \eta_n^* \quad \rightarrow \quad \eta_n = \pm 1 \quad (\text{hermitian field})\,. \tag{8.97}$$

and fields are accordingly categorized as

	$\eta = +1$	$\eta = -1$
spin − 0	$\mathcal{P}\phi(x)\mathcal{P}^\dagger = \phi(Px)$ scalar	$\mathcal{P}\phi(x)\mathcal{P}^\dagger = -\phi(Px)$ pseudo-scalar
spin − 1	$\mathcal{P}A_\mu(x)\mathcal{P}^\dagger = -A^\mu(Px)$ axial vector	$\mathcal{P}A_\mu(x)\mathcal{P}^\dagger = A^\mu(Px)$ vector

(8.98)

Note that for spin-1 fields, the intrinsic parity is given by the sign of the transformation of the space part. For charged, or non-hermitian, fields, η is not by itself restricted to ± 1; however, when charged particles are related to neutral particles by some symmetries such as isospin, then it is often natural to assign the same intricsic parity to the charged particle as that of the neutral counter part. What is dictating the choice, however, is if one can make the given interaction commute with $\mathcal{P}$.

8.2.4 Choice of parity phases and parity invariance

We have seen that the parity operator $\mathcal{P}$ can be made to commute with the QED interaction Hamitltonian h_{QED} by picking the intrinsic parity of photon to be -1, and this indicated that all processes caused by QED are symmetric under parity inversion. Then, question is when one cannot choose intrinsic parities so that the parity operator commutes with interaction Hamiltonian.

If the photon field hypothetically couples to an axial current

$$\mathcal{H}_{\rm int}(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\,A_\mu(x)\,, \tag{8.99}$$

then all the derivation stays the same as before except in (8.82), we have

$$\gamma^0\gamma_\mu\gamma^0 = \gamma^\mu \quad \to \quad \gamma^0\gamma_\mu\gamma_5\gamma^0 = -\gamma^\mu\gamma_5\,, \tag{8.100}$$

generating an overall minus sign. This makes the current to transform as an axial vector field:

$$\mathcal{P}j^A_\mu(x)\mathcal{P}^\dagger = -j^{A\mu}(x')\,, \tag{8.101}$$

with $x' \equiv Px$ and

$$j^A_\mu(x) \stackrel{\text{def}}{\equiv} \bar\psi(x)\gamma_\mu\gamma_5\psi(x)\,. \tag{8.102}$$

In this case, the parity of the photon can be chosen to be +1 to make the interaction commute with $\mathcal{P}$. Thus, the physics represented by the interaction $\bar\psi\gamma^\mu\gamma_5\psi A_\mu$ by itself is also symmetric under parity.

Then, what if the photon couples to the vector and axial currents?

$$\begin{aligned}\mathcal{H}_{\text{int}}(x) &= a\,\bar\psi(x)\gamma^\mu\gamma_5\psi(x)\,A_\mu(x) + b\,\bar\psi(x)\gamma^\mu\psi(x)\,A_\mu(x)\\ &= \bar\psi(x)\gamma^\mu(a+b\gamma_5)\psi(x)\,A_\mu(x)\,,\end{aligned} \tag{8.103}$$

where a and b are any non-zero constants. Then in order for this interaction to satisfy $\mathcal{P}\mathcal{H}_{\text{int}}(x)\mathcal{P} = \mathcal{H}_{\text{int}}(x')$, the intrinsic parity of the photon needs to be taken as −1 for the first term and +1 for the second term. However, for a given $\mathcal{P}$ operator and the given particle, the intrinsic parity can be chosen only once, and thus $\mathcal{P}$ cannot be made to commute with this interaction Hamiltonian.

If we take $a = 1$ and $b = -1$, the interaction is the $V - A$ interaction we have seen for the weak interaction except that the vector field here is a hermitian field and the fermion current is also hermitian. In fact, the interaction Hamiotonian can be written as

$$\mathcal{H}_{\text{int}}(x) = 2\bar\psi(x)\gamma_\mu P_L\psi(x)\,, \tag{8.104}$$

with $P_L = (1-\gamma_5)/2$ is the projection operator for the left-handed chirality. Then, as we have seen in (5.158) and below, the fermions created by this interaction will be left-handed and anti-fermions will be right-handed in the high-energy limit. Such processes is clearly not symmetric under parity inversion.

The situation is similar for Yukawa-type couplings. We have seen in (8.74) that the Yukawa interaction term $h_Y = \int d^3x\bar\psi\psi\phi$ can be made to commute with $\mathcal{P}$ by taking the intrinsic parity of the scalar to be +1. On the other hand, if the fermion current is a pseudo-scalar, then using the transformation of fields (8.96), we have

$$\begin{aligned}\mathcal{P}\,\bar\psi(x)\gamma_5\psi(x)\,\mathcal{P}^\dagger &= \underbrace{\eta_f\eta_f^*}_{1}\,\bar\psi(x')\,\underbrace{\gamma^0\gamma_5\gamma^0}_{-\gamma_5}\,\psi(x')\\ &= -\bar\psi(x')\gamma_5\psi(x') \quad (x' \equiv Px)\,,\end{aligned} \tag{8.105}$$

where η_f is the intrinsic parity of the fermion. Then, by taking the intrinsic parity of the spin-0 particle η_s to be -1, we can accomplish

$$\begin{aligned} \mathcal{P}\,\bar{\psi}(x)\gamma_5\psi(x)\,\phi(x)\,\mathcal{P}^\dagger &= -\eta_s^*\bar{\psi}(x')\gamma_5\psi(x')\,\phi(x') \\ &= \bar{\psi}(x')\gamma_5\psi(x')\,\phi(x') \quad (\eta_s = -1)\,; \end{aligned} \tag{8.106}$$

namely, the interaction Hamiltonian now commutes with $\mathcal{P}$. And since the parity phases could be chosen for this to happen, the physics represented by $\mathcal{H}_{\rm int} = \bar{\psi}\gamma_5\psi\,\phi$ is symmetric under parity. If the spin-0 particle couples to a scalar current and a pseudoscalar current simultaneously, however, one cannot choose the intrinsic parity of the spin-0 particle to make the entire interaction Hamiltonian commute with $\mathcal{P}$. In particular, it is easy to see that in the processes caused by the interaction

$$\mathcal{H}_{\rm int} = \bar{\psi}(1-\gamma_5)\psi\,\phi = 2\,\bar{\psi}P_L\psi\,\phi\,, \tag{8.107}$$

fermions and antifermions are right-handed if they are in the final state (in the high-energy limit), and are left-handed if they are in the initial state. The decay $s \to f\bar{f}$, for example, the final state where both f and $\bar{f}$ are right-handed will turn into the state where both are left-handed under parity, two final states should have the same rate if the physics were symmetric under parity. Such would be the case if the interaction were purely scalar or purely pseudoscalar, but not if they are mixed.

Exercise 8.2 *Show that the fermions or antifermions created by the interaction (8.107) are right-handed (in the high energy limit), and that only the left-handed componenets of fermions or antifermions in the initial state contribute to interaction.*

Then, you may ask whether there is any difference between the pure scalar coupling and the pure pseudoscalar coupling. Actually there is. Let us evaluate the dedcay rate $s \to f(p)\bar{f}(q)$ in both cases. The vertex factor is simply $-i$ for the scalar coupling and $-i\gamma_5$ for the pseudoscalar coupling. Then the Lorentz-invariant matrix element is

$$\mathcal{M} = \begin{cases} -i\bar{u}_p v_q & (\bar{\psi}\psi) \\ -i\bar{u}_p\gamma_5 v_q & (\bar{\psi}\gamma_5\psi) \end{cases}\,. \tag{8.108}$$

Squaring this and summing over the final-state spins, the resulting decay rate is

$$\Gamma = \frac{|\vec{p}|}{4\pi} \times \begin{cases} \beta^2 & (\bar{\psi}\psi) \\ 1 & (\bar{\psi}\gamma_5\psi) \end{cases}\,, \tag{8.109}$$

where $\vec{p}$ is the momentum of the fermion in the C.M. system of the decay, and β is its velocity. Thus, the decay rate for the scalar coupling is suppressed by factor β^2 with respect to the pseudoscalar case, which would be expected if the orbital angular momentum of the final state is $l = 1$ (P wave) for the scalar coupling and $l = 0$ (S wave) for the pseudoscalar coupling. This suppression is sometimes called the centrifugal-force effect.

Exercise 8.3 *Take the matrix elements of (8.108), square them, sum over spins and obtain the decay rates (8.109).*

The above argument did not involve any intrinsic parities. In fact, all information about decay rates, relations among rates and selection rules etc., are included in the Lorentz-invariant matrix elements which are obtained by the Feynman rules without any recourse to the parity phases. However, one can see that the scalar case is suppressed by the centrifugal-force effect just by the parity phase argument without resorting to actual evaluation of the rates. First, we note that if we choose the intrinsic parity of the spin-0 particle to be $+1$, then the scaler interaction commutes with $\mathcal{P}$ and thus one can use the conservation of the parity eigenvalue; namely, the parity of the final state created by the scalar interaction should also be $+1$ for the same parity operator $\mathcal{P}$ for which the intrinsic parity of the spin-0 particle is taken to be $+1$. Similarly, if we choose the the intrinsic parity of the spin-0 particle to be -1, then the pseudoscalar interaction conserves parity and thus the final state created thereby would have parity -1 for that parity operator.

Now, as we will see below, the parity eigenvalue of a two-body state consisting of particles 1 and 2 is given by

$$\boxed{\mathcal{P}|l,m;1,2\rangle = (-)^l \eta_1\eta_2\,|l,m;1,2\rangle}\,. \tag{8.110}$$

where $\eta_{1,2}$ are the intrinsic parities of the two particles and where l is the absolute value and m is the component along z axis of the orbital angular momentum. The two particles are understood to have fixed spin components along z axis. Then for the $f\bar{f}$ final state, $\eta_{\bar{f}} = -\eta_f^*$ gives $\eta_f\eta_{\bar{f}} = -\eta_f\eta_f^* = -1$, and thus the parity eigenvalue is -1 for S wave or any l-even state, and $+1$ for P wave or any l-odd state. Thus, the scalar coupling should result in a final state with odd l and the pseudoscalar coupling with even l. Since in general lower l-values dominate rates unless other selection rules comes in to play, we see that the scalar coupling creates mostly P wave and the pseudoscalar coupling mostly S wave. Thus, if the coupling constants are the same, one expects that the scalar case to be suppressed by the centrifugal-force effect with respect to the pseudoscalar case.

Now let's prove the transformation property under $\mathcal{P}$ (8.110) for a two-body state with definite orbital angular momentum $|l,m\rangle$. Such state is formed by first combining particles 1 and 2 in a back-to-back motion as $|(1,\vec{p},\sigma_1)(2,-\vec{p},\sigma_2)\rangle$ and integrating over all direction with the spherical harmonics wave function $Y^l_m(\hat{p})$ where $\hat{p}$ represents the polar angles (θ,ϕ) of the direction $\hat{p}$:

$$|l,m;1,2\rangle \equiv \int d^3\hat{p}\; Y^l_m(\hat{p})\,|(1,\vec{p},\sigma_1)(2,-\vec{p},\sigma_2)\rangle\,, \tag{8.111}$$

where $d^3\hat{p}$ is the angle element of the direction of $\hat{p}$, namely $\hat{p} \equiv d\phi\, d\cos\theta$, and $|l,m;1,2\rangle$ indicates that it is a state with orbital angular momentum (l,m) formed by particles 1 and 2. The spin indices $\sigma_{1,2}$ are the components along z axis as before. Applying $\mathcal{P}$ to this state and using (8.21),

$$\begin{aligned}
\mathcal{P}|l,m;1,2\rangle &= \int d^3\hat{p}\, Y^l_m(\hat{p})\, \mathcal{P}|(1,\vec{p},\sigma_1)(2,-\vec{p},\sigma_2)\rangle \\
&= \eta_1\eta_2 \int d^3\hat{p}\, Y^l_m(\hat{p})\, |(1,-\vec{p},\sigma_1)(2,\vec{p},\sigma_2)\rangle \\
(\vec{p}\to-\vec{p}) &= \eta_1\eta_2 \int d^3\hat{p}\, \underbrace{Y^l_m(-\hat{p})}_{(-)^l Y^l_m(\hat{p})}\, |(1,\vec{p},\sigma_1)(2,-\vec{p},\sigma_2)\rangle \\
&= (-)^l \eta_1\eta_2\, |l,m;1,2\rangle\,,
\end{aligned} \tag{8.112}$$

where we have used the property of the spherical harmonics

$$Y^l_m(-\hat{p}) = (-)^l Y^l_m(\hat{p})\,. \tag{8.113}$$

Thus, the parity of the two-body state in orbital angular momentum l is the product of the intrinsic parities of the particles times $(-)^l$ proving (8.110).

Similarly, one could ask how to tell that the photon-fermion coupling is a pure vector coupling and not an axial vector coupling? This, one knows by the fact that all known processes caused by QED are symmetric under parity inversion, which tells us that the coupling is not a mixture of vector and axial vector, and the threshold behavior of $e^+e^-\to f\bar{f}$ which rises as $\propto|\vec{p}|$, where $\vec{p}$ is the fermion momentum in the C.M. system, and not as $|\vec{p}|\beta^2$, which selects pure vector over pure axial vector.

Incidentally, one of the most stringent test of P-invariance of QED is the electric dipole moment of electron:

$$d(\text{electron}) < (0.3\pm0.8)\times10^{-26}\ (e\cdot\text{cm})\,. \tag{8.114}$$

Since the electric dipole moment is measured against the electron spin which is the only intrinsic reference of direction, the electric dipole moment changes sign under parity inversion - the spin stays the same and the charge distribution is inverted. Thus, existence of an intrinsic electric dipole moment measured with respect to spin would signal parity violation. Is the number above small or big? A natural scale of length is the classical electron radius $r_e=\alpha/m\sim2.8\times10^{-13}$cm (6.243) which is the smallest of the three length scales of QED. If charges e and $-e$ are separated by r_e, the electric dipole moment would be $2.8\times10^{-13}\,e{\cdot}$cm which is 10^{13} times larger than the experimental upper limit.

So far, we have dealt with interactions with explicit form. For some interactions such as the strong interaction of hadrons, exact form of interaction is difficult to write down. We then approach it by assigning intrinsic parities to particles such that all known processes that are believed to be caused by the given interaction conserve parity. When one finds a process that does not allow the conservation of parity eigenvalues, one would either declare that the parity is not conserved by the interaction in question, or that some other interaction that violates parity is contributing to the process.

The decay of the ρ^0 meson

$$\rho^0 \to \pi^+\pi^- \tag{8.115}$$

occurs through the strong interaction which conserves parity. The ρ^0 meson is spin-1 and $\pi^\pm$ are spinless; thus, the final state is in the $l = 1$ state to conserve the angular momentum. The parity of the final state is then

$$\eta(\pi^+\pi^-) = \underbrace{(-)^l}_{-1}\underbrace{\eta_{\pi^+}\eta_{\pi^-}}_{1} = -1 \tag{8.116}$$

where we have used the fact that π^- is the antiparticle of π^+ and the relation $\eta_n^* = \eta_{\bar{n}}$ for spin-0 particles. The observation of this process thus indicates that the parity of ρ^0 is -1. Namely, the ρ^0 meson is a vector particle (as opposed to axial vector particle) according to (8.98). On the other hand, there exists a spin-1 particle called a_1 which cannot decay to $\pi^+\pi^-$:

$$a_1^0 \not\to \pi^+\pi^- \,. \tag{8.117}$$

Thus, the absence of this decay strongly suggests that the parity of a_1 is $+1$; a_1 is an axial vector particle. Then, in the observed decay

$$a_1^0 \to \rho^0\pi^0 \,, \tag{8.118}$$

the orbital angular momentum of the final state should be even, or predominantly S wave, since the parity of π^0 is known to be -1 from the analysis of an electromagnetic decay $\pi^0 \to e^+e^-e^+e^-$ (the double Dalitz decay). The S wave nature of the decay was experimentally verified by measuring the angular distribution of the decay $\rho^0 \to \pi^+\pi^-$.

8.3 Charge conjugation

The treatment of charge conjugation, or particle-antiparticle exchange, follows the similar line of logic as that of parity. We begin by defining the charge conjugation operator in the Hilbert space.

8.3.1 Definition of the $\mathcal{C}$ operator

We define the charge conjugation operator $\mathcal{C}$ to be the one that transforms $a^\dagger_{n\,\vec{p}\,\sigma}$ to $a^\dagger_{\bar{n}\,\vec{p}\,\sigma}$ and vice versa; namely, exchanges particle and antiparticle without changing momentum and spin:

$$\boxed{\mathcal{C}a^\dagger_{n\,\vec{p}\,\sigma}\mathcal{C}^\dagger = \xi_{n\,\vec{p}\,\sigma}a^\dagger_{\bar{n}\,\vec{p}\,\sigma}} \tag{8.119}$$

where the phase factor $\xi_{n\,\vec{p}\,\sigma}$ in general depends on the particle type, momentum, and spin. Note that a particle and its antiparticle are treated as separate particles; namely, the above definition includes (with $\bar{\bar{n}} = n$)

$$\mathcal{C}a^\dagger_{\bar{n}\,\vec{p}\,\sigma}\mathcal{C}^\dagger = \xi_{\bar{n}\,\vec{p}\,\sigma}a^\dagger_{n\,\vec{p}\,\sigma}\,. \tag{8.120}$$

As in the case of the $\mathcal{P}$ operator, the set of phases $\{\xi_{n\,\vec{p}\,\sigma}\}$ completely determines the $\mathcal{C}$ operator in the Hilbert space and is shown to be unitary:

$$\mathcal{C}^\dagger\mathcal{C} = 1\,. \tag{8.121}$$

Again, similarly to $\mathcal{P}$, the overall phase of $\mathcal{C}$ is defined by requiring that the vacuum is invariant under $\mathcal{C}$:

$$\mathcal{C}|0\rangle = |0\rangle\,, \qquad \mathcal{C}^\dagger|0\rangle = |0\rangle\,, \tag{8.122}$$

where the second relation is obtained by applying $\mathcal{C}^\dagger$ to the both sides of the first. Then, applying (8.119) to the vacuum, one obtains

$$\mathcal{C}|n\,\vec{p}\,\sigma\rangle = \xi_{n\,\vec{p}\,\sigma}|\bar{n}\,\vec{p}\,\sigma\rangle\,. \tag{8.123}$$

A multi-particle states transform with a multiplicative phase factor

$$\begin{aligned}
\mathcal{C}|\{n_i\,\vec{p}_i\,\sigma_i\}\rangle &= \mathcal{C}a^\dagger_{n_1\,\vec{p}_1\,\sigma_1}\underbrace{}_{\mathcal{C}^\dagger\mathcal{C}}\cdots\underbrace{}_{\mathcal{C}^\dagger\mathcal{C}}a^\dagger_{n_k\,\vec{p}_k\,\sigma_k}\underbrace{}_{\mathcal{C}^\dagger\mathcal{C}}|0\rangle \\
&= \xi_{n_1\,\vec{p}_1\,\sigma_1}\cdots\xi_{n_k\,\vec{p}_k\,\sigma_k}a^\dagger_{\bar{n}_1\,\vec{p}_1\,\sigma_1}\cdots a^\dagger_{\bar{n}_k\,\vec{p}_k\,\sigma_k}|0\rangle\,,
\end{aligned} \tag{8.124}$$

where the normalization factor is implicit which differs from unity when there are more than one particle with identical quantum numbers. Thus,

$$\mathcal{C}|\{n_i\,\vec{p}_i\,\sigma_i\}\rangle = \xi_{\{n_i\,\vec{p}_i\,\sigma_i\}}|\{\bar{n}_i\,\vec{p}_i\,\sigma_i\}\rangle\,, \tag{8.125}$$

with

$$\xi_{\{n_i\,\vec{p}_i\,\sigma_i\}} \equiv \xi_{n_1\,\vec{p}_1\,\sigma_1}\cdots\xi_{n_k\,\vec{p}_k\,\sigma_k}\,. \tag{8.126}$$

We will now apply this $\mathcal{C}$ operator to the Yukawa interaction and the QED interaction and pick the charge conjugation phases $\xi_{n\,\vec{p}\,\sigma}$ such that these interactions commute with $\mathcal{C}$ up to normal ordering:

$$:\mathcal{C}\mathcal{H}_{\rm int}(x)\mathcal{C}^\dagger: \;=\; :\mathcal{H}_{\rm int}(x): \tag{8.127}$$

with

$$\mathcal{H}_{\rm int}(x) = \bar{\psi}(x)\psi(x)\,\phi(x) \quad \text{or} \quad \bar{\psi}(x)\gamma_\mu\psi(x)\,A^\mu(x)\,. \tag{8.128}$$

The normal ordering is legitimate since as long as $\mathcal{C}\mathcal{H}_{\rm int}(x)\mathcal{C}^\dagger$ and $\mathcal{H}_{\rm int}(x)$ are the same after normal ordering, they results in the same physics. Hereafter, the normal ordering is implicit when we refer to commuatation of $\mathcal{H}_{\rm int}$ and $\mathcal{C}$. Note that in the case of charge conjugation, the space-time variable x is the same on the both sides of the equality, and the Hamiltonian density itself should commute with $\mathcal{C}$; namely, no need to integrate over $\vec{x}$ to make the interaction Hamiltonian $h(t)$ commute with the transformation operator as in the case of $\mathcal{P}$. If we could choose $\xi_{n\,\vec{p}\sigma}$'s such that $\mathcal{H}_{\rm int}$ commutes with $\mathcal{C}$, then the processes caused by the interaction are symmetric under particle-antiparticle exchange. To check this, we must first find the transformation properties of fields under $\mathcal{C}$.

8.3.2 Transformation of fields under $\mathcal{C}$

For a spin-0 field, we drop the spin index σ, and write (8.120) and the hermitian conjugate of (8.119) as

$$\mathcal{C}a^\dagger_{\bar{s}\,\vec{p}}\mathcal{C}^\dagger = \xi_{\bar{s}\,\vec{p}}\,a^\dagger_{s\,\vec{p}}\,, \qquad \mathcal{C}a_{s\,\vec{p}}\mathcal{C}^\dagger = \xi^*_{s\,\vec{p}}\,a_{\bar{s}\,\vec{p}} \tag{8.129}$$

where $n = s$ indicates that these are for a spin-0 particle. Then, applying $\mathcal{C}$ to $\phi(x) = \sum_{\vec{p}}\left(a_{s\vec{p}}\,e_{\vec{p}}(x) + a^\dagger_{\bar{s}\vec{p}}\,e^*_{\vec{p}}(x)\right)$ and using the above, we obtain

$$\begin{aligned}\mathcal{C}\phi(x)\mathcal{C}^\dagger &= \sum_{\vec{p}}\left(\mathcal{C}a_{s\,\vec{p}}\mathcal{C}^\dagger\,e_{\vec{p}}(x) + \mathcal{C}a^\dagger_{\bar{s}\,\vec{p}}\mathcal{C}^\dagger\,e^*_{\vec{p}}(x)\right)\\ &= \sum_{\vec{p}}\left(\xi^*_{s\,\vec{p}}\,a_{\bar{s}\,\vec{p}}\,e_{\vec{p}}(x) + \xi_{\bar{s}\,\vec{p}}\,a^\dagger_{s\,\vec{p}}\,e^*_{\vec{p}}(x)\right).\end{aligned} \tag{8.130}$$

We could leave it as it is, but as in the case of $\mathcal{P}$, if $\phi(x)$ does not transform to something proportional to $\phi(x)$, or possibly $\phi^\dagger(x)$, then no phase adjustments on other fields can make the interaction commute with $\mathcal{C}$. The right hand side of (8.130) contains $a_{\bar{s}\,\vec{p}}$ and $a^\dagger_{s,\vec{p}}$ while $\phi(x)$ contains $a_{s\,\vec{p}}$ and $a^\dagger_{\bar{s},\vec{p}}$; thus, it cannot be proportional to $\phi(x)$, but it can be proportional to $\phi^\dagger(x)$:

$$\mathcal{C}\phi(x)\mathcal{C}^\dagger = (\text{const})\phi^\dagger(x) = (\text{const})\sum_{\vec{p}}\left(a^\dagger_{s\vec{p}}\,e^*_{\vec{p}}(x) + a_{\bar{s}\vec{p}}\,e_{\vec{p}}(x)\right). \tag{8.131}$$

The condition for $\mathcal{C}\phi(x)\mathcal{C}^\dagger$ to be written in this form are: the phases $\xi_{s\,\vec{p}}$ and $\xi_{\bar{s}\,\vec{p}}$ should not depend on momentum:

$$\xi_{s\,\vec{p}} = \xi_s\,, \qquad \xi_{\bar{s}\,\vec{p}} = \xi_{\bar{s}}\,, \tag{8.132}$$

and the charge-conjugation phase of the antiparticle is the complex conjugate of that of the particle:

$$\xi_{\bar{s}} = \xi^*_s\,. \tag{8.133}$$

Then, the transformation of a spin-0 field is

$$\mathcal{C}\phi(x)\mathcal{C}^\dagger = \xi_s^* \phi^\dagger(x)\,. \tag{8.134}$$

Thus, choosing the charge-conjugation phases reduces to assigning a phase for the spin-0 particle. For a self-conjugate particle, $\xi_{\bar{s}} = \xi_s^*$ means that ξ_s is real, namely ± 1. Setting $\bar{s} = s$ in (8.123), we see that ξ_s is the eigenvalue of the $\mathcal{C}$ operator for a single-particle state with any momentum:

$$\mathcal{C}|s\,\vec{p}\rangle = \xi_s |s\,\vec{p}\rangle \qquad (s = \bar{s})\,. \tag{8.135}$$

Such ξ_s for a self-conjugate field is called the intrinsic charge parity of the particle.

If you are not comfortable with the argument above that lead to $\xi_{s\,\vec{p}} = \xi_s$, $\xi_{\bar{s}\,\vec{p}} = \xi_{\bar{s}}$, and $\xi_{\bar{s}} = \xi_s^*$, just read on; you will see that we have not overly constrained ourselves to make the interactions commute with $\mathcal{C}$. Or, you may use (8.130) directly in the requirement that the interaction commutes with $\mathcal{C}$ where the above conditions will result after all.

The transformation of a fermion field under $\mathcal{C}$ is a little more involved. First, we write (8.120) and the hermitian conjugate of (8.119) as

$$\mathcal{C} a^\dagger_{\bar{f}\vec{p}\sigma} \mathcal{C}^\dagger = \xi_{\bar{f}\vec{p}\sigma} a^\dagger_{f\vec{p}\sigma}\,, \qquad \mathcal{C} a_{f\vec{p}\sigma} \mathcal{C}^\dagger = \xi^*_{f\vec{p}\sigma}\, a_{\bar{f}\vec{p}\sigma}\,, \tag{8.136}$$

where we have set $n = f$ for fermions. Applying $\mathcal{C}$ to the momentum expansion $\psi(x) = \sum_{\vec{p},\sigma} \Big(a_{f\vec{p}\sigma} f_{\vec{p}\sigma}(x) + a^\dagger_{\bar{f}\vec{p}\sigma} g_{\vec{p}\sigma}(x) \Big)$, we obtain

$$\begin{aligned} \mathcal{C}\psi(x)\mathcal{C}^\dagger &= \sum_{\vec{p},\sigma} \Big(\mathcal{C} a_{f\vec{p}\sigma} \mathcal{C}^\dagger f_{\vec{p}\sigma}(x) + \mathcal{C} a^\dagger_{\bar{f}\vec{p}\sigma} \mathcal{C}^\dagger g_{\vec{p}\sigma}(x) \Big) \\ &= \sum_{\vec{p},\sigma} \Big(\xi^*_{f\vec{p}\sigma}\, a_{\bar{f}\vec{p}\sigma} f_{\vec{p}\sigma}(x) + \xi_{\bar{f}\vec{p}\sigma}\, a^\dagger_{f\vec{p}\sigma} g_{\vec{p}\sigma}(x) \Big) \end{aligned} \tag{8.137}$$

As in the case of $\mathcal{P}$, we would like this to be something of the form (const. matrix)$\times\psi(x)$, but it is not possible since $\mathcal{C}\psi(x)\mathcal{C}^\dagger$ contains $a_{\bar{f}\vec{p}\sigma}$ and $a^\dagger_{f\vec{p}\sigma}$ while $\psi(x)$ contains $a_{f\vec{p}\sigma}$ and $a^\dagger_{\bar{f}\vec{p}\sigma}$. It could be 'proportional' to $\psi^\dagger(x)$ which contains the same creation and annihilation operators as $\mathcal{C}\psi(x)\mathcal{C}^\dagger$; however, this encounters difficulty since $\mathcal{C}\psi(x)\mathcal{C}^\dagger$ is a column vector while $\psi^\dagger(x)$ is a row vector. Thus, we define a simple conjugation of a column vector without changing it to a row vector:

$$w \equiv \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \quad \to \quad w^* \equiv \begin{pmatrix} w_1^* \\ w_2^* \\ w_3^* \\ w_4^* \end{pmatrix} \qquad (w_i\text{: complex number})\,, \tag{8.138}$$

and

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \to \quad \psi^* \equiv \begin{pmatrix} \psi_1^\dagger \\ \psi_2^\dagger \\ \psi_3^\dagger \\ \psi_4^\dagger \end{pmatrix} \qquad (\psi_i\text{: operator})\,. \tag{8.139}$$

For a matrix, the complex conjugation is defined similarly by simply taking conjugate of each element:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \quad \to \quad A^* = \begin{pmatrix} A^*_{11} & \cdots & A^*_{1n} \\ \vdots & \ddots & \vdots \\ A^*_{n1} & \cdots & A^*_{nn} \end{pmatrix}. \tag{8.140}$$

Thus, we will try to choose the charge-conjugation phases such that

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = (\text{const. matrix})\,\psi^*(x)\,, \tag{8.141}$$

where

$$\psi^*(x) = \sum_{\vec{p},\sigma} \Big(a^\dagger_{f\vec{p}\sigma} f^*_{\vec{p}\sigma}(x) + a_{\bar{f}\vec{p}\sigma} g^*_{\vec{p}\sigma}(x) \Big)\,. \tag{8.142}$$

Comparing this with (8.137), we see that we need a relation between $f_{\vec{p}\sigma}$ and $g^*_{\vec{p}\sigma}$ or equivalently, that between $u_{\vec{p}\sigma}$ and $v^*_{\vec{p}\sigma}$. Now we will show that there exists a 4×4 matrix Γ which is unitary and symmetric

$$\boxed{\Gamma^\dagger \Gamma = 1\,, \quad \Gamma^T = \Gamma}\,, \tag{8.143}$$

and satisfies

$$\boxed{\Gamma \gamma^*_\mu \Gamma^\dagger = -\gamma_\mu \quad (\mu = 0, 1, 2, 3)}\,, \tag{8.144}$$

and using such matrix Γ, we can write

$$\boxed{v_{\vec{p}\sigma} = \Gamma u^*_{\vec{p}\sigma}\,, \quad u_{\vec{p}\sigma} = \Gamma v^*_{\vec{p}\sigma}}\,. \tag{8.145}$$

Before proving (8.145), let's see if a matrix Γ that satisfies (8.143) and (8.144) exists for an arbitrary representation of gamma matrices. In the Dirac representation, all γ_μ are real except γ_2 which is purely imaginary. Thus, the condition (8.144) above means that Γ should commute with γ_2 and anticommute with γ_μ ($\mu \neq 2$). The matrix γ_2 itself fits the bill and it is also unitary and symmetric. Thus,

$$\Gamma = e^{i\phi}\gamma_2 \qquad (\text{Dirac rep.})\,, \tag{8.146}$$

where the phase factor $e^{i\phi}$ can be adjusted to match the relative phase of u and v spinors so that (8.145) holds, and the exact value is not important for what follows.

Suppose now that Γ is unitary and symmetric and satisfies (8.144). In a different representation, the gamma matrices γ'_μ are related to the ones in the Dirac representation by a unitary transformation V as discussed in Section 3.5:

$$\gamma'_\mu = V\gamma_\mu V^\dagger\,. \tag{8.147}$$

Then, it is straightforward to show that a 4×4 matrix Γ' defined by

$$\Gamma' = V\Gamma V^T \tag{8.148}$$

is unitary, symmetric, and satisfies (8.144) for γ'_μ. Thus, such matrix Γ exists for any arbitrary representation.

Exercise 8.4 *Representation independence of* Γ.
Suppose a 4×4 *matrix* Γ *is unitary and symmetric and satisfies* $\Gamma\gamma^*_\mu\Gamma^\dagger = -\gamma_\mu$. *Show that* $\Gamma' \equiv V\Gamma V^T$ *is unitary, symmetric, and satisfies*

$$\Gamma'\gamma'^*_\mu\Gamma'^\dagger = -\gamma'_\mu \quad (\mu = 0,1,2,3)\,, \tag{8.149}$$

where V *is an arbitrary* 4×4 *unitary matrix and* $\gamma'_\mu = V\gamma_\mu V^\dagger$.

To prove (8.145), we recall that $u_{\vec p\sigma}$ and $v_{\vec p\sigma}$ ($\sigma = \pm 1/2$) are eigenspinors of the energy sign operator $\not p/m$ and the spin operator $\gamma_5\not s$ and form a complete orthonormal set, and thus the eigen values of the two operators determine the spinors up to an overall constant. Thus, we start from the relation

$$(\not p - m)u_{\vec p\sigma} = 0\,, \quad \gamma_5\not s u_{\vec p\sigma} = s_\sigma\, u_{\vec p\sigma}\,, \tag{8.150}$$

where

$$s_\sigma \stackrel{\text{def}}{\equiv} \operatorname{sign}(\sigma)\,. \tag{8.151}$$

Taking the complex conjugate of the first, and using (8.144) and $\Gamma^\dagger\Gamma = 1$, we get

$$\underbrace{\;}_{\Gamma}(\gamma^*_\mu p^\mu - m)\underbrace{\;}_{\Gamma^\dagger\Gamma}u^*_{\vec p\sigma} = 0 \quad\rightarrow\quad (\underbrace{\Gamma\gamma^*_\mu\Gamma^\dagger}_{-\gamma_\mu \text{ by } (8.144)} p^\mu - m)\Gamma u^*_{\vec p\sigma} = 0\,, \tag{8.152}$$

thus, $\Gamma u^*_{\vec p\sigma}$ has $\not p/m = -1$:

$$(\not p + m)\Gamma u^*_{\vec p\sigma} = 0\,. \tag{8.153}$$

Taking the complex conjugate of $\gamma_5\not s u_{\vec p\sigma} = s_\sigma\, u_{\vec p\sigma}$,

$$\underbrace{\;}_{\Gamma}\gamma^*_5\gamma^*_\mu s^\mu\underbrace{\;}_{\Gamma^\dagger\Gamma}u^*_{\vec p\sigma} = s_\sigma \underbrace{\;}_{\Gamma}u^*_{\vec p\sigma} \quad\rightarrow\quad (\Gamma\gamma^*_5\gamma^*_\mu\Gamma^\dagger)s^\mu\Gamma u^*_{\vec p\sigma} = s_\sigma\,\Gamma u^*_{\vec p\sigma}\,. \tag{8.154}$$

Now, (8.144) leads to

$$\Gamma(\overbrace{\gamma_{\mu_1}}^{}\cdots\overbrace{\gamma_{\mu_k}}^{})^*\Gamma^\dagger = (-)^k\gamma_{\mu_1}\cdots\gamma_{\mu_k}\,. \qquad (8.155)$$

$$\Gamma^\dagger\Gamma \quad \Gamma^\dagger\Gamma$$

Then, we have

$$\Gamma\gamma_5^*\gamma_\mu^*\Gamma^\dagger = -i\Gamma(\gamma^0\gamma^1\gamma^2\gamma^3\gamma_\mu)^*\Gamma^\dagger = -i(-)^5\gamma^0\gamma^1\gamma^2\gamma^3\gamma_\mu = \gamma_5\gamma_\mu\,. \qquad (8.156)$$

The equation (8.154) then shows that $\Gamma u_{\vec{p}\sigma}$ is the eigenspinor of $\gamma_5\not{s}$ with eigenvalues s_σ:

$$\underbrace{(\Gamma\gamma_5^*\gamma_\mu^*\Gamma^\dagger)}_{\gamma_5\gamma_\mu} s^\mu\Gamma u^*_{\vec{p}\sigma} = s_\sigma\,\Gamma u^*_{\vec{p},\pm\vec{s}} \quad\rightarrow\quad \gamma_5\not{s}\Gamma u^*_{\vec{p}\sigma} = s_\sigma\Gamma u^*_{\vec{p}\sigma}\,. \qquad (8.157)$$

Also, $\Gamma u^*_{\vec{p}\sigma}$ is properly normalized:

$$(\Gamma u^*_{\vec{p}\sigma})^\dagger\Gamma u^*_{\vec{p}\sigma} = u^T_{\vec{p}\sigma}\underbrace{\Gamma^\dagger\Gamma}_{1}u^*_{\vec{p}\sigma} = (u^\dagger_{\vec{p}\sigma}u_{\vec{p}\sigma})^T = 2E\,. \qquad (8.158)$$

Thus, $\Gamma u^*_{\vec{p}\sigma}$ should be equal to $v_{\vec{p}\sigma}$ up to a phase which can be eliminated by adjusting the phase of Γ, namely $v_{\vec{p}\sigma} = \Gamma u^*_{\vec{p}\sigma}$ which is the first of the relations (8.145). From the unitarity condition $\Gamma^\dagger\Gamma = 1$ and $\Gamma^T = \Gamma$,

$$\Gamma^\dagger\Gamma = 1 \quad\xrightarrow{T}\quad \underbrace{\Gamma^T}_{\Gamma}\Gamma^* = 1 \quad\rightarrow\quad \Gamma\Gamma^* = 1\,. \qquad (8.159)$$

Taking the complex conjugate of $v_{\vec{p}\sigma} = \Gamma u^*_{\vec{p}\sigma}$, we then have

$$v^*_{\vec{p}\sigma} = \Gamma^* u_{\vec{p}\sigma} \quad\xrightarrow{\times\Gamma}\quad \Gamma v^*_{\vec{p}\sigma} = \underbrace{\Gamma\Gamma^*}_{1}u_{\vec{p}\sigma}\,, \qquad (8.160)$$

which proves the second of (8.145). The relations (8.145) can be written in terms of $f_{\vec{p}\sigma}$ and $g_{\vec{p}\sigma}$ as

$$g_{\vec{p}\sigma}(x) = \Gamma f^*_{\vec{p}\sigma}(x)\,, \quad f_{\vec{p}\sigma}(x) = \Gamma g^*_{\vec{p}\sigma}(x)\,. \qquad (8.161)$$

Since $g_{\vec{p}\sigma}(x)$ and $f_{\vec{p}\sigma}(x)$ are solutions of the Dirac equation, so are $\Gamma f^*_{\vec{p}\sigma}(x)$ and $\Gamma g^*_{\vec{p}\sigma}(x)$.

Using (8.161), the transformation of $\psi(x)$ (8.137) becomes

$$\begin{aligned}\mathcal{C}\psi(x)\mathcal{C}^\dagger &= \sum_{\vec{p},\sigma}\Big(\xi^*_{f\vec{p}\sigma}\,a_{\bar{f}\vec{p}\sigma}\Gamma g^*_{\vec{p}\sigma}(x) + \xi_{\bar{f}\vec{p}\sigma}\,a^\dagger_{f\vec{p}\sigma}\Gamma f^*_{\vec{p}\sigma}(x)\Big)\\ &= \Gamma\sum_{\vec{p},\sigma}\Big(\xi^*_{\bar{f}\vec{p}\sigma}\,a_{f\vec{p}\sigma}f_{\vec{p}\sigma}(x) + \xi_{f\vec{p}\sigma}\,a^\dagger_{\bar{f}\vec{p}\sigma}g_{\vec{p}\sigma}(x)\Big)^* \qquad (8.162)\end{aligned}$$

In order for this to be written in the form (8.141), we should have

$$\xi_{f\vec{p}\sigma} = \xi_f \,, \qquad \xi_{\bar{f}\vec{p}\sigma} = \xi_{\bar{f}} \,, \tag{8.163}$$

and

$$\xi_{\bar{f}} = \xi_f^* \,. \tag{8.164}$$

Then, the transformation (8.162) is now written as

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \xi_f^* \,\Gamma\psi^*(x) \,. \tag{8.165}$$

The action of $\mathcal{C}$ on a spin-1 field is the same as for a spin-0 field. The transformation of creation and annihilation operators are

$$\mathcal{C}a^\dagger_{\bar{v}\,\vec{p}\sigma}\mathcal{C}^\dagger = \xi_{\bar{v}\,\vec{p}\sigma}a^\dagger_{v\,\vec{p}\sigma} \,, \qquad \mathcal{C}a_{v\,\vec{p}\sigma}\mathcal{C}^\dagger = \xi^*_{v\,\vec{p}\sigma}\,a_{\bar{v}\,\vec{p}\sigma} \,, \tag{8.166}$$

where we have set $n = v$ to indicate that they are for a spin-1 particle. Then, $A^\mu(x)$ transforms as

$$\begin{aligned}
\mathcal{C}A^\mu(x)\mathcal{C}^\dagger &= \sum_{\vec{p},\sigma}\Big(\mathcal{C}a_{v\,\vec{p}\sigma}\mathcal{C}^\dagger\epsilon^\mu_{\vec{p}\sigma}e_{\vec{p}}(x) + \mathcal{C}a^\dagger_{\bar{v}\,\vec{p}\sigma}\mathcal{C}^\dagger\epsilon^{\mu*}_{\vec{p}\sigma}e^*_{\vec{p}}(x)\Big) \\
&= \sum_{\vec{p},\sigma}\Big(\xi^*_{v\,\vec{p}\sigma}\,a_{\bar{v}\,\vec{p}\sigma}\epsilon^\mu_{\vec{p}\sigma}e_{\vec{p}}(x) + \xi_{\bar{v}\,\vec{p}\sigma}a^\dagger_{v\,\vec{p}\sigma}\epsilon^{\mu*}_{\vec{p}\sigma}e^*_{\vec{p}}(x)\Big) \\
&= \sum_{\vec{p},\sigma}\Big(\xi^*_{\bar{v}\,\vec{p}\sigma}a_{v\,\vec{p}\sigma}\epsilon^\mu_{\vec{p}\sigma}e_{\vec{p}}(x) + \xi_{v\,\vec{p}\sigma}\,a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^{\mu*}_{\vec{p}\sigma}e^*_{\vec{p}}(x)\Big)^\dagger .
\end{aligned} \tag{8.167}$$

Again, this cannot be 'proportional to' $A^\mu(x)$ since it has wrong creation and annihilation operators. However, it can be proportional to $A^{\mu\dagger}(x)$ if the phases do not depend on $\vec{p}$ and σ:

$$\xi_{v\,\vec{p}\sigma} = \xi_v \,, \qquad \xi_{\bar{v}\,\vec{p}\sigma} = \xi_{\bar{v}} \,, \tag{8.168}$$

and

$$\xi_{\bar{v}} = \xi_v^* \,. \tag{8.169}$$

Then, the transformation of A^μ can now be written as

$$\mathcal{C}A^\mu(x)\mathcal{C}^\dagger = \xi_v^* \,A^{\mu\dagger}(x) \,. \tag{8.170}$$

As in the case of spin-0 fields, if A^μ is hermitian, or equivalently the particle is self-conjugate, then the relation $\xi_{\bar{v}} = \xi_v^*$ leads to $\xi_v = \pm 1$ which becomes an eigenvalue of the operator $\mathcal{C}$ for any single-particle state $|v\,\vec{p}\sigma\rangle$; namely, ξ_v is the intrinsic charge parity of the self-conjugate spin-1 particle.

Now, let's apply all this to the Yukawa coupling $\mathcal{H}_{\rm int} = \bar{\psi}\psi\,\phi$ to see how it works. Taking the hermitian conjugate of $\mathcal{C}\psi\mathcal{C}^\dagger = \xi_f^*\,\Gamma\psi^*$ (8.165), we get

$$\mathcal{C}\psi^\dagger(x)\mathcal{C}^\dagger = \xi_f\,\psi^T\Gamma^\dagger \,. \tag{8.171}$$

Then, the scalar current transforms as

$$\begin{aligned} :\mathcal{C}\,\bar{\psi}\psi\,\mathcal{C}^\dagger: &= \; :\mathcal{C}\,\overbrace{\psi^\dagger}^{\mathcal{C}^\dagger}\;\gamma^0\;\overbrace{\psi\,\mathcal{C}^\dagger}^{\mathcal{C}}: \\ &= \; :(\xi_f\,\psi^T\Gamma^\dagger)\gamma^0(\xi_f^*\,\Gamma\psi^*): \\ &= \; \underbrace{\xi_f\xi_f^*}_{1}:\psi^T(\Gamma^\dagger\gamma^0\Gamma)\,\psi^*: \;. \end{aligned} \tag{8.172}$$

Now, multiplying (8.155) by $\Gamma^\dagger$ from the left and Γ from the right,

$$(\gamma_{\mu_1}\cdots\gamma_{\mu_k})^* = (-)^k\Gamma^\dagger\,\gamma_{\mu_1}\cdots\gamma_{\mu_k}\,\Gamma\,. \tag{8.173}$$

Using $\gamma_\mu^\dagger = \gamma^\mu$, we have

$$\begin{aligned} \Gamma^\dagger\,\gamma_{\mu_1}\cdots\gamma_{\mu_k}\,\Gamma &= (-)^k\,(\gamma_{\mu_1}\cdots\gamma_{\mu_k})^* \\ &= (-)^k\,[(\gamma_{\mu_1}\cdots\gamma_{\mu_k})^\dagger]^T \\ &= (-)^k[\gamma_{\mu_k}^\dagger\cdots\gamma_{\mu_1}^\dagger]^T \\ &= (-)^k[\gamma^{\mu_k}\cdots\gamma^{\mu_1}]^T\,, \end{aligned} \tag{8.174}$$

where the ordering of γ_μ's is inverted and the Lorentz indices changed from superscripts to subscripts. Applying this to γ^0 gives

$$\Gamma^\dagger\gamma^0\Gamma = -\gamma_0^T\,, \tag{8.175}$$

which can be obtained more directly, but (8.174) will be useful later. Then, $:\mathcal{C}\,\bar{\psi}\psi\,\mathcal{C}^\dagger:$ is

$$\begin{aligned} :\mathcal{C}\,\bar{\psi}\psi\,\mathcal{C}^\dagger: &= \; -:\psi^T\gamma_0^T\psi^*: = -:\psi_i\,(\gamma_0)_{ji}\,\psi_j^\dagger: \\ &= \; :\psi_j^\dagger\,(\gamma_0)_{ji}\,\psi_i: \\ &= \; :\bar{\psi}\psi:, \end{aligned} \tag{8.176}$$

where we have used

$$:A\,B: = \; -:B\,A:, \tag{8.177}$$

for A and B are any linear combinations of fermion creation and annihilation operators. Using $\mathcal{C}\phi\mathcal{C}^\dagger = \xi_s\phi^\dagger$, the Yukawa interaction Hamiltonian transforms as

$$\begin{aligned} :\mathcal{C}\,\bar{\psi}\psi\,\phi\,\mathcal{C}: &= \; :\mathcal{C}\bar{\psi}\psi\,\mathcal{C}^\dagger: \;\mathcal{C}\phi\mathcal{C}^\dagger \\ &= \; \xi_s^*:\bar{\psi}\psi\,\phi: \;. \end{aligned} \tag{8.178}$$

Thus, the Yukawa interaction commutes with $\mathcal{C}$ (up to normal ordering) by taking the charge parity of the spin-0 particle to be

$$\xi_s = +1\,, \tag{8.179}$$

and, since we could pick such phase, all processes caused by the Yukawa interaction $\bar{\psi}\psi\,\phi$ is symmetric under particle-antiparticle exchange.

If the spin-0 field couples to a pseudoscalar current as $\mathcal{H}_{\rm int} = \bar{\psi}\gamma_5\psi\,\phi$, then it is straightforward to show that the interaction Hamiltonian also commutes with $\mathcal{C}$ with the choice $\xi_s = +1$. The derivation is similar to the above and is left as an exercise. Thus, all processes caused by $\mathcal{H}_{\rm int} = \bar{\psi}\gamma_5\psi\,\phi$ are also invariant under charge conjugation. Since the choice $\xi_s = +1$ works for both the scalar and pseudoscalar couplings, $\mathcal{C}$ commutes with any linear combination of the two:

$$\mathcal{H}_{\rm int} = \bar{\psi}(a + b\gamma_5)\psi\,\phi\,, \tag{8.180}$$

where a and b are any complex constants. Namely, all processes caused by this interaction are symmtric under particle-antiparticle exchange.

Exercise 8.5 *Pseudoscaler coupling under $\mathcal{C}$.*
(*a*) *Apply (8.174) to show*

$$\Gamma^\dagger\gamma_0\gamma_5\Gamma = -(\gamma_0\gamma_5)^T\,. \tag{8.181}$$

(*b*) *Prove that $\mathcal{H}_{\rm int} = \bar{\psi}\gamma_5\psi\,\phi$ commutes with $\mathcal{C}$ with $\xi_s = +1$; namely,*

$$\mathcal{C}\,\bar{\psi}\gamma_5\psi\,\phi\,\mathcal{C}^\dagger = \bar{\psi}\gamma_5\psi\,\phi\,. \tag{8.182}$$

Normal ordering is implicit.

Let us now look at the transformation of the QED interaction Hamiltonian under charge conjugation. Applying $\mathcal{C}$ to the QED Hamiltonian density $\mathcal{H}_{\rm int} = j_\mu A^\mu$, we have

$$\mathcal{C}\mathcal{H}_{\rm int}\mathcal{C}^\dagger = \mathcal{C}j_\mu\mathcal{C}^\dagger\,\mathcal{C}A^\mu\mathcal{C}^\dagger = \mathcal{C}j_\mu\mathcal{C}^\dagger\,\xi_v^* A^\mu\,, \tag{8.183}$$

where we have used the transformation of a spin-1 field (8.170) with $A^\dagger_\mu = A_\mu$ for photon. Using (8.165) and its Hermitian conjugate (8.171), $j_\mu \equiv \bar{\psi}\gamma^\mu\psi$ transforms as

$$\begin{aligned}
\mathcal{C}j_\mu\mathcal{C}^\dagger &= \mathcal{C}\,\underbrace{\psi^\dagger}_{\mathcal{C}^\dagger}\gamma_0\gamma_\mu\underbrace{\psi\,\mathcal{C}^\dagger}_{\mathcal{C}} \\
&= (\xi_f\psi^T\Gamma^\dagger)\gamma_0\gamma_\mu(\xi_f^*\Gamma\psi^*) \\
&= \psi^T\Gamma^\dagger\gamma_0\gamma_\mu\Gamma\psi^*
\end{aligned} \tag{8.184}$$

On the other hand, (8.174) with $\gamma_{\mu_1} = \gamma_0$ and $\gamma_{\mu_2} = \gamma_\mu$ gives

$$\Gamma^\dagger\gamma_0\gamma_\mu\Gamma = (\gamma^\mu\gamma^0)^T = (\gamma_0\gamma_\mu)^T\,. \tag{8.185}$$

Writing the normal ordering explicitly, and noting that one can flip the ordering of two fermion fields inside a normal ordering provided that a minus sign is added as in (8.177),

$$\begin{aligned} :\mathcal{C} j_\mu \mathcal{C}^\dagger: &= :\psi^T (\gamma_0\gamma_\mu)^T \psi^* : = : \psi_i \, (\gamma_0\gamma_\mu)_{ji} \, \psi_j^\dagger : \\ &= - : \psi_j^\dagger \, (\gamma_0\gamma_\mu)_{ji} \, \psi_i : = - : \psi^\dagger \gamma_0 \gamma_\mu \psi : \\ &= - : j_\mu : \, . \end{aligned} \tag{8.186}$$

Namely, both the space part and the time part of the current change sign under charge conjugation, which is expected when one changes the sign of the charge carried by the particle. This is independent of the charge phase of the fermion ξ_f which cancelled in (8.184). The transformation of the QED Hamiltonian (8.183) is now

$$\mathcal{C}\mathcal{H}_{\rm int}\mathcal{C}^\dagger = -\xi_v^* \, j_\mu A^\mu = -\xi_v^* \mathcal{H}_{\rm int} \, ; \tag{8.187}$$

thus, the QED Hamiltonian commutes with $\mathcal{C}$ provided that we take the intricsic charge parity of photon to be -1:

$$\xi_v = -1 \qquad (QED) \, . \tag{8.188}$$

Since we could choose the charge phases such that the $\mathcal{H}_{\rm int}$ commutes with $\mathcal{C}$, all processes caused by QED are symmetric under particle-antiparticle exchange.

In summary, the transformation properties of the fields under $\mathcal{C}$ are

$$\boxed{\begin{array}{rcll} \mathcal{C}\phi(x)\mathcal{C}^\dagger &=& \xi_n^* \phi^\dagger(x) & (\text{spin} - 0) \\ \mathcal{C}\psi(x)\mathcal{C}^\dagger &=& \xi_n^* \Gamma \psi^*(x) & (\text{spin} - \frac{1}{2}) \\ \mathcal{C}A_\mu(x)\mathcal{C}^\dagger &=& \xi_n^* A_\mu^\dagger(x) & (\text{spin} - 1) \\ \xi_{\bar{n}} &=& \xi_n^* & (\text{spin} - 0, \frac{1}{2}, 1) \end{array}} \, , \tag{8.189}$$

where Γ is a unitary and symmetric matrix that satisfies $\Gamma\gamma_\mu^*\Gamma^\dagger = -\gamma_\mu$. Note that the defining equation for the charge conjugation phase

$$\mathcal{C}|n \, \vec{p} \, \sigma\rangle = \xi_n |n \, \vec{p} \, \sigma\rangle \tag{8.190}$$

is not an eigenvalue equation unless the state is self-conjugate, namely $n = \bar{n}$. This is in contrast to the case of parity where the eigenvalue of $\mathcal{P}$ exists for any particle as long as the particle is at rest. For a self-conjugate particle, the intrinsic charge parity is an eigenvalue of $\mathcal{C}$ and should be either $+1$ or -1. According to (8.188), the intrinsic charge parity of photon is -1 which came about essentially because the charge current j^μ changes sign under $\mathcal{C}$, and in order to keep $\mathcal{H}_{\rm int}$ invariant, A^μ also had to change sign under $\mathcal{C}$.

8.3.3 C invariance and selection rules

Suppose the interaction Hamiltonian is invariant under $\mathcal{C}$

$$\mathcal{C}\mathcal{H}_{\rm int}\mathcal{C}^\dagger = \mathcal{H}_{\rm int}\,. \tag{8.191}$$

Since $\mathcal{C}$ does not affect time ordering, the S operator is then also invariant under $\mathcal{C}$ [see (8.28)]:

$$\mathcal{C}S\mathcal{C}^\dagger = S\,. \tag{8.192}$$

Now if the initial and final states are eigenstates of $\mathcal{C}$

$$\mathcal{C}|i\rangle = \xi_i|i\rangle\,, \quad \mathcal{C}|f\rangle = \xi_f|f\rangle\,, \tag{8.193}$$

then we have just as in the case of the parity case (8.27)

$$\langle f|S|i\rangle = \langle f|\mathcal{C}^\dagger\mathcal{C}S\mathcal{C}^\dagger\mathcal{C}|i'\rangle = \xi_f^*\xi_i\langle f|S|i\rangle\,. \tag{8.194}$$

Namely, the transition amplitude is zero unless $\xi_f^*\xi_i = 1$ or, multiplying ξ_f on both sides

$$\xi_i = \xi_f\,. \tag{8.195}$$

Again, we are assuming that the charge conjugation phases of have been chosen such that the interaction Hamiltonian became invariant under $\mathcal{C}$.

The situation is thus essentially the same as in the parity case. One difference is that the parity eigenstates were particles at rest or some special kinematic configurations as in (8.110) because all momenta flipped sign under parity, while the intrinsic charge parity is well-defined as long as all particles are self-conjugate. In fact, setting $\bar{n}_i = n_i$ in (8.28) and noting that the charge phases in general do not depend on momentum and spin of the particle ($\xi_{n\vec{p}\sigma} = \xi_n$), we have

$$\mathcal{C}|\{n_i\,\vec{p}_i\,\sigma_i\}\rangle = \xi_{n_1}\cdots\xi_{n_k}|\{n_i\,\vec{p}_i\,\sigma_i\}\rangle \quad \text{(all particles self-conjugate)}\,; \tag{8.196}$$

namely, the charge parity of a state consisting of only self-conjugate particles are the product of the intrinsic charge parities of the individual particles regardless of momenta or spins. For example, we have seen in (8.188) that the intrinsic charge parity of photon has to be chosen to be -1 in order for the QED interaction Hamiltonian to be invariant under $\mathcal{C}$. Then, if a state contains n photons and nothing else, then it is an eigenstate of $\mathcal{C}$ with eigenvalue $(-1)^n$. Then, $\xi_i = \xi_f$ immediately indicates

$$\text{(even number of photons)} \quad \not\leftrightarrow \quad \text{(odd number of photons)}\,. \tag{8.197}$$

This is a special case of Furry's theorem which is valid even if the photons are off-shell. The neutral pion decays predominantly to two photons

$$\pi^0 \to 2\gamma \tag{8.198}$$

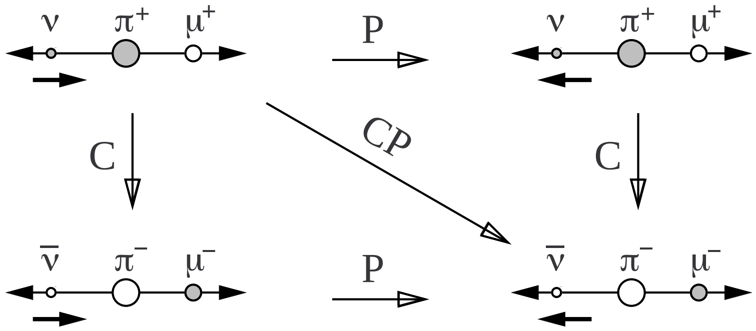


Figure 8.2: The $\pi^+ \to \mu^+\nu$ decay under $\mathcal{C}$ and $\mathcal{P}$.

through the electromagnetic interaction and the strong interaction both of which are believed to conserve C. Thus, we assign the charge parity of π^0 to be $+1$. Then, the decay of π^0 to odd number of photons should be prohibited. In fact,

$$Br(\pi^0 \to 3\gamma) < 3.1 \times 10^{-8}\,, \tag{8.199}$$

where 'Br' indicates the branching fraction, which is a good confirmation of the C-invariance of the electromagnetic interaction which is believed to be mainly responsible for these decays.

8.4 The CKM matrix and CP violation

At the high-energy limit, the $V-A$ interaction creates left-handed fermions or right-handed antifermions. Under particle-antiparticle exchange, the statement becomes 'the $V-A$ interaction creates left-handed antifermions and right-handed fermions' which contradicts the original statement. Thus, the $V-A$ interaction is not invariant under charge conjugation. However, if one further transforms the above statement by parity, it becomes 'the $V-A$ interaction creates right-handed antifermions and left-handed fermions' which is the same as the first statement. Thus, one expects that the $V-A$ interaction is invariant under $\mathcal{C}$ followed by $\mathcal{P}$, or $\mathcal{CP}$. It should not matter which is applied first.

For example, the neutrino in the decay $\pi^+ \to \mu^+\nu$ (Figure 8.2) is purely left-handed assuming that the neutrino is massless. Since π^+ is spinless and since the orbital angular momentum of two-body decay cannot have a component along the back-to-back flight direction in the C.M. system, the spin of μ^+ is also left-handed even though it is an antifermion which is possible since μ^+ is massive. It is besides the point, however, and we will focus on the spin of the neutrino. Under parity, or mirror inversion, the spin of the neutrino becomes right-handed keeping all particle types the same. Such decay is not allowed for the $V - A$ interaction. However, if one further apply particle-antiparticle exchange to it, then the antineutrino is right-handed, and such process is allowed; in fact, the decay rate of $\pi^- \rightarrow \mu^-\bar{\nu}$ where $\bar{\nu}$ is right-handed is the same as that of $\pi^+ \rightarrow \mu^+\nu$ where ν is left-handed both theoretically and experimentally.

Thus, it appears that the weak $V - A$ interaction is invariant under CP. As we will see, however, the quark-W coupling in the standard model, where there are three pairs of quarks coupling to W, is in general not invariant under CP. In fact, this is believed to be the source of the CP violating effects observed in the neutral kaon system. There are two eigenstates of mass and decay rate for the neutral kaon system: K_S (lifetime $\sim 0.89\times10^{-10}$ sec) and K_L (lifetime $\sim 5.2\times10^{-8}$ sec). One of the observed CP violating effects is that the longer-lived component K_L decays more to final states containing e^+ than to those containing e^- with an asymmetry of about 3×10^{-3}. Why is this a CP violation? Consider a creation of K_L state by a proton-antiproton ($p\bar{p}$) annilation at rest. Then the phenomenon observed is that the rate of $p\bar{p} \rightarrow e^+$ is greater than the rate of $p\bar{p} \rightarrow e^-$ where $e^\pm$ came from something with the mass of K_L and long-lived. Under particle-antiparticle exchange (C), this statement becomes 'the rate of $p\bar{p} \rightarrow e^-$ is greater than the rate of $p\bar{p} \rightarrow e^+$ where $e^\pm$ came from something with the mass of K_L and long-lived', which is in contradiction with the original statement. Thus, this phenomena violates C. Under the parity inversion, this latter statement does not change since there is no specification of handedness in the statement. Thus, the phenomenon also violates CP.

Such CP-violating couplings or related physics may be partially responsible for the baryon number excess of our universe. [1] Assuming that the universe was originally matter-antimatter symmetric, there are three conditions for baryon number excess to occur:

1. Baryon-number-violating interactions. If all interactions conserve baryon number, of course there will be no baryon number excess.

2. Both C and CP are violated. If C is conserved, then creation of matter and antimatter will occur at the same rate. Since P does not change the total count of matter or antimatter, if CP is conserved there will be no matter-antimatter asymmetry created either.

3. Non-adiabatic processes. Even if baryon number is violated and C or CP is not conserved, if reactions and their inverse reactions occur at the same rate (i.e.

[1] A baryon consists of three quarks and proton and neutron are examples. The baryon number of a proton or a neutron is +1 and that of an antiproton or an antineutron is −1. A quark carries a baryon number +1/3.

the reaction is adiabatic) then there will be no net change in the total baryon number.

Since C is known to be badly violated in the weak interaction, the relevant question usually is whether CP is violated or not.

If indeed the quark-W coupling of the standard model is responsible for the CP violation in the neutral kaon system, then variety of large CP violating effects are expected in the neutral B meson system where a B meson is a meson that contains a b quark. Even though detailed treatment of these effects is beyond the scope of this book, we will at least demonstrate that the CP symmetry can be violated in the standard model in the context of quantum field theory.

8.4.1 CP invariance of the $e\nu W$ coupling

Let us apply some of the transformation of fields obtained earlier to the $V-A$ coupling of W to $e\nu$:

$$h_{e\nu W}(t) = \int d^3x \mathcal{H}_{e\nu W}(x) \; + \; \int d^3x \mathcal{H}^\dagger_{e\nu W}(x)\,, \tag{8.200}$$

where

$$\mathcal{H}_{e\nu W} = \bar\nu\gamma_\mu(1-\gamma_5)e\,W^\mu\,. \tag{8.201}$$

The neutrino is actually the electron nutrino ν_e, but we will drop the subscript e for simplicity. The Hermitian conjugate of $\mathcal{H}_{e\nu W}$ is given by

$$\mathcal{H}^\dagger_{e\nu W} = \bar e\gamma_\mu(1-\gamma_5)\nu\,W^{\mu\dagger}\,. \tag{8.202}$$

We have ignored the real coupling constant for simplicity. Applying $\mathcal{CP}$ to $\mathcal{H}_{e\nu W}$,

$$\mathcal{CP}\mathcal{H}_{e\nu W}\mathcal{P}^\dagger\mathcal{C}^\dagger = \mathcal{CP}\,\bar\nu\gamma_\mu(1-\gamma_5)e\,\mathcal{P}^\dagger\mathcal{C}^\dagger\;\mathcal{CP}\,W^\mu\,\mathcal{P}^\dagger\mathcal{C}^\dagger\,. \tag{8.203}$$

As we have seen in (8.96) and (8.189), the W^μ field transforms under $\mathcal{C}$ and $\mathcal{P}$ as

$$\mathcal{P}W^\mu(x)\mathcal{P}^\dagger = -\eta_W^* W_\mu(x') \quad (x'\equiv Px)\,, \quad \mathcal{C}W^\mu(x)\mathcal{C}^\dagger = \xi_W^* W^{\mu\dagger}(x)\,. \tag{8.204}$$

Thus, the W field transforms under $\mathcal{CP}$ as

$$\mathcal{CP}W^\mu(x)\mathcal{P}^\dagger\mathcal{C}^\dagger = -\eta_W^*\xi_W^* W_\mu^\dagger(x') \tag{8.205}$$

with $x' = Px$. The transformation of a vector and an axial-vector current under $\mathcal{P}$ was already obtained in (8.82) and (8.101). The only difference that we now have two different fermions forming the currents, and thus the fermion parity phases do not cancel out:

$$\begin{aligned}
\mathcal{P}\,\bar\nu(x)\gamma_\mu e(x)\,\mathcal{P}^\dagger &= \eta_\nu\eta_e^*\,\bar\nu(x')\gamma^\mu e(x') \\
\mathcal{P}\,\bar\nu(x)\gamma_\mu\gamma_5 e(x)\,\mathcal{P}^\dagger &= -\eta_\nu\eta_e^*\,\bar\nu(x')\gamma^\mu\gamma_5 e(x')\,.
\end{aligned} \tag{8.206}$$

Thus, the $e\nu$ $V - A$ current transforms under $\mathcal{P}$ as

$$\mathcal{P}\,\bar{\nu}(x)\gamma_\mu(1-\gamma_5)e(x)\,\mathcal{P}^\dagger = \eta_\nu\eta_e^*\,\bar{\nu}(x')\gamma^\mu(1+\gamma_5)e(x')\,. \tag{8.207}$$

It changed $(1-\gamma_5)$ to $(1+\gamma_5)$, γ_μ (subscript) to γ^μ (superscript), and x to $x' = Px$.

The transformation of the $e\nu$ vector current is derived similarly to that of the QED current. The difference again is that we have now two different fermion fields, and (8.186) now becomes

$$\begin{aligned} :\mathcal{C}\bar{\nu}\gamma_\mu e\,\mathcal{C}^\dagger: &= \xi_\nu\xi_e^* :\nu^T(\gamma_0\gamma_\mu)^T e^*: = \xi_\nu\xi_e^* :\nu_i\,(\gamma_0\gamma_\mu)_{ji}\,e_j^\dagger: \\ &= -\xi_\nu\xi_e^* :e_j^\dagger\,(\gamma_0\gamma_\mu)_{ji}\,\nu_i: = -\xi_\nu\xi_e^* :\bar{e}\gamma_\mu\nu:\ . \end{aligned} \tag{8.208}$$

The derivation of the transformation of the axial current is similar and straightforward and is left as an exercise:

$$:\mathcal{C}\bar{\nu}\gamma_\mu\gamma_5 e\,\mathcal{C}^\dagger := \xi_\nu\xi_e^* :\bar{e}\gamma_\mu\gamma_5\nu:\ . \tag{8.209}$$

Exercise 8.6 *Use the relation (8.174) to prove (8.209).*

Using (8.209) and (8.208), the transformation of (8.207) under $\mathcal{C}$ is then

$$\begin{aligned} \mathcal{CP}\,\bar{\nu}(x)\gamma_\mu(1-\gamma_5)e(x)\,\mathcal{P}^\dagger\mathcal{C}^\dagger &= \eta_\nu\eta_e^*\,\mathcal{C}\,\bar{\nu}(x')\gamma^\mu(1+\gamma_5)e(x')\,\mathcal{C}^\dagger \\ &= -(\eta_\nu\xi_\nu)(\eta_e\xi_e)^*\,\bar{e}(x')\gamma^\mu(1-\gamma_5)\nu(x')\,, \end{aligned} \tag{8.210}$$

where we have dropped the implicit normal ordering. Note that the Lorentz index μ changed from subscript to superscript and that left-handed coupling is restored even though the position of e and ν are interchanged. Combining this and the transformation of W under $\mathcal{CP}$ (8.205), the transformation of $\mathcal{H}_{e\nu W}$ under $\mathcal{CP}$ (8.203) now becomes

$$\begin{aligned} \mathcal{CP}\mathcal{H}_{e\nu W}(x)\mathcal{P}^\dagger\mathcal{C}^\dagger &= (\eta_\nu\xi_\nu)(\eta_e\xi_e)^*(\eta_W\xi_W)^*\,\bar{e}(x')\gamma^\mu(1-\gamma_5)\nu(x')\,W_\mu^\dagger(x') \\ &= (\eta_\nu\xi_\nu)(\eta_e\xi_e)^*(\eta_W\xi_W)^*\,\mathcal{H}_{e\nu W}^\dagger(x')\,, \end{aligned} \tag{8.211}$$

where the change of the Lorentz index on γ_μ from subscript to superscript is compensated by that of W from superscipt to subscript, and we have used (8.202). Then, taking the phase factors to be

$$(\eta_\nu\xi_\nu)(\eta_e\xi_e)^*(\eta_W\xi_W)^* = 1\,, \tag{8.212}$$

we have

$$\mathcal{CP}\,\mathcal{H}_{e\nu W}(x)\mathcal{P}^\dagger\mathcal{C}^\dagger = \mathcal{H}_{e\nu W}(x') \qquad (x' = Px)\,. \tag{8.213}$$

Integrating both sides over space, we obtain

$$\mathcal{CP}\int d^3x\,\mathcal{H}_{e\nu W}(x)\,\mathcal{P}^\dagger\mathcal{C}^\dagger = \int d^3x\,\mathcal{H}_{e\nu W}^\dagger(x)\,, \tag{8.214}$$

where the dummy integration variable of the right hand side was changed from x' to x. The Hermitian conjugate of this is

$$\mathcal{CP}\int d^4x\,\mathcal{H}^\dagger_{e\nu W}(x)\,\mathcal{P}^\dagger\mathcal{C}^\dagger = \int d^4x\,\mathcal{H}_{e\nu W}(x)\,. \tag{8.215}$$

Namely, with the choice of phases given by (8.212), the first term and the second term of the interaction Hamiltonian $h_{e\nu W}(t)$ (8.200) are simply exchanged under $\mathcal{CP}$. As the result, the whole $h_{e\nu W}(t)$ commutes with $\mathcal{CP}$:

$$\mathcal{CP}h_{e\nu W}(t)\mathcal{P}^\dagger\mathcal{C}^\dagger = h_{e\nu W}(t)\,. \tag{8.216}$$

Since we could chose the $\mathcal{C}$ and $\mathcal{P}$ phases such that $h(t)$ commutes with $\mathcal{CP}$, we now know that all processes caused by this interaction are symmetric under CP.

Similarly, the $\mu\nu_\mu W$ coupling as well as the $\tau\nu_\tau W$ coupling are invariant under CP, and thus the entire three 'generations' of lepton-neutrino-W couping (5.243) is invariant under CP. Namely, the interaction Hamiltonian density

$$\mathcal{H}_{\ell\nu W} = \frac{g}{2\sqrt{2}}\bar\nu_i\gamma_\mu(1-\gamma_5)\ell_i\,W^\mu \quad (i=1,2,3\text{ summed}) \tag{8.217}$$

with

$$(\nu_1,\nu_2,\nu_3)\equiv(\nu_e,\nu_\mu,\nu_\tau)\,,\qquad (\ell_1,\ell_2,\ell_3)\equiv(e,\mu,\tau)\,, \tag{8.218}$$

satisfies

$$\mathcal{CP}\,\mathcal{H}_{\ell\nu W}(x)\,\mathcal{P}^\dagger\mathcal{C}^\dagger = \mathcal{H}^\dagger_{\ell\nu W}(x')\qquad (x'=Px)\,, \tag{8.219}$$

when the $\mathcal{C}$ and $\mathcal{P}$ phases are chosen such that

$$(\eta_{\nu_i}\xi_{\nu_i})(\eta_{\ell_i}\xi_{\ell_i})^*(\eta_W\xi_W)^* = 1 \quad (i=1,2,3)\,. \tag{8.220}$$

Then, the Hamitotonian

$$h_{\ell\nu W}(t) = \int d^3x\,(\mathcal{H}_{\ell\nu W}(x)+\mathcal{H}^\dagger_{\ell\nu W}(x)) \tag{8.221}$$

will commute with $\mathcal{CP}$. Note however that we have assumed there will be no $\ell\nu W$ couplings that change the lepton generation; namely, no terms such as $\mu\nu_e W$ and $\tau\nu_\mu W$. If such terms exist, then the overall $\ell\nu W$ interaction may not commute with $\mathcal{CP}$ for the same reason why the quark-W interaction can violate CP as we will see below.

The decay $\pi^+\to\mu^+\nu_\mu$ mentioned earlier is caused by the u and $\bar d$ quark pair that constitutes the π^+ is annihilated by a quark-W coupling and the resulting W turns into μ^+ and ν_μ by the CP-conserving $\mu\nu_\mu W$ coupling (Figure 8.3). As we will see below, the quark-W coupling as a whole does not in general commute with $\mathcal{CP}$. However, it does not mean that all processes caused by the interaction should violate CP. As it turns out, CP violating effects usually require inteference of more than two diagrams, and the CP violation in the π^+ decay, which is dominated by the single diagram, happens to be negligible.

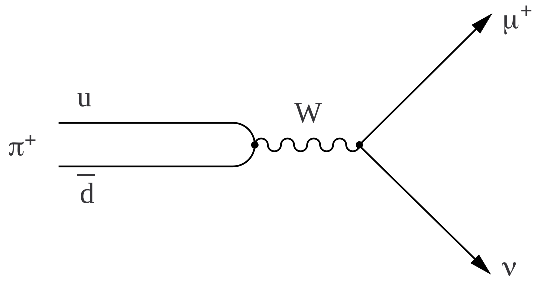


Figure 8.3: The diagram for the $\pi^+ \to \mu^+\nu_\mu$ decay. The π^+ meson contains a u quark and a $\bar{d}$ quark which are annihilated by a quark-W coupling. The internediate W then materializes as a $\mu^+\nu_\mu$ pair.

8.4.2 The quark-W coupling and CP violation

Now we will examine the quark-W coupling and show that when there are more than two generations of quarks it could violate CP. The quark-W coupling in the standard model is given by

$$h_{qW}(t) = \int d^3x\, (\mathcal{H}_{qW}(x) + \mathcal{H}^\dagger_{qW}(x)) \tag{8.222}$$

with

$$\mathcal{H}_{qW} = V_{ij}\, \bar{u}_i\gamma_\mu(1-\gamma_5)d_j\; W^\mu \quad (i,j = 1,2,3 \text{ summed}) \tag{8.223}$$

where

$$(u_1,u_2,u_3) \equiv (u,c,t)\,, \qquad (d_1,d_2,d_3) \equiv (d,s,b)\,, \tag{8.224}$$

and V is the Cabbibo-Kobayashi-Masukawa (CKM) matrix which is theoretically exptected to be unitary:

$$V^\dagger V = 1\,. \tag{8.225}$$

We have ignored the overall real coupling constant $g/2\sqrt{2}$. We will derive this form of quark-W coupling later based on the spontaneously broken gauge symmetry; for now, however, we will accept it as given.

Applying the CP transformation of the $e\nu$ current (8.210) to the quark current, we have

$$\begin{aligned}\mathcal{CP}\; \bar{u}_i(x)\gamma_\mu(1-\gamma_5)d_j(x)\; \mathcal{P}^\dagger\mathcal{C}^\dagger \; &= \\ -(\eta_{u_i}\xi_{u_i})(\eta_{d_j}\xi_{d_j})^*\; \bar{d}_j(x')\gamma^\mu(1-\gamma_5)u_i(x') \qquad &(x' = Px)\,.\end{aligned} \tag{8.226}$$

Together with the CP transformation of W^μ (8.205), $\mathcal{H}_{qW}$ then transforms as

$$\mathcal{CP}\,\mathcal{H}_{qW}(x)\,\mathcal{P}^\dagger\mathcal{C}^\dagger = \alpha_{u_i}\alpha^*_{d_j}\alpha^*_W\, V_{ij}\, \bar{d}_j(x')\gamma^\mu(1-\gamma_5)u_i(x')\, W^\dagger_\mu(x')\,, \tag{8.227}$$

where

$$\alpha_x \equiv \eta_x \xi_x \qquad (x = u_i, d_j, \text{etc.}) \tag{8.228}$$

is the CP phase factor that can be choosen arbitrarily for each particle type. On the other hand, the Hermitian conjugate of $\mathcal{H}_{qW}$ is

$$\mathcal{H}^\dagger_{qW} = V^*_{ij}\, \bar{d}_j \gamma_\mu (1-\gamma_5) u_i\, W^{\mu\dagger} \,. \tag{8.229}$$

Then, if we can choose the CP phases such that

$$\alpha_{u_i} \alpha^*_{d_j} \alpha^*_W\, V_{ij} = V^*_{ij} \tag{8.230}$$

for all (i, j), then we have

$$\mathcal{CP}\, \mathcal{H}_{qW}(x)\, \mathcal{P}^\dagger \mathcal{C}^\dagger = \mathcal{H}^\dagger_{qW}(x') \,, \tag{8.231}$$

and h_{qW} will commute with the $\mathcal{CP}$ operator upon integration over space.

Can we choose the arbitrary CP phases α_x's so that (8.230) holds? If we can, then processes caused by the quark-W coupling are symmetric under CP. Since α_W in (8.230) is just an overall phase, it is redundant; for example, it can be replaced by the overall CP phase of the d-type quarks. Thus, we can set α_W to 1 and the criteria is equivalent to

$$e^{i(\phi_{u_i} - \phi_{d_j})} = \frac{V^*_{ij}}{V_{ij}} = e^{-2i\theta_{ij}} \qquad (\phi_x \equiv \arg \alpha_x,\ \theta_{ij} \equiv \arg V_{ij}) \,, \tag{8.232}$$

or

$$\phi_{u_i} - \phi_{d_j} = -2\theta_{ij} \qquad (\mathrm{mod}\, 2\pi) \,. \tag{8.233}$$

For a given 3×3 unitary matrix V, there are 9 equations corresponding to θ_{ij} $(i, j = 1, 2, 3)$. There are 6 CP phases (ϕ_{u_i}, ϕ_{d_i}) $(i, j = 1, 2, 3)$ to choose, but in reality there are only 5 degrees of freedom since only relative phases appear in (8.233). Does that mean that we are $9 - 5 = 4$ degrees of freedom short to satisfy the criteria? Not so quickly. Since V is unitary, not all θ_{ij}'s are independent. What we need is the number of degree of freedom associated with phases. There are 9 complex numbers in V which have 18 degrees of freedom. The unitarity condition $V^\dagger V = 1$ has 9 equations which are in general complex. However, since $V^\dagger V$ is Hermitian for any matrix V, we can write

$$V^\dagger V = \begin{pmatrix} r_{11} & c_{12} & c_{13} \\ c^*_{12} & r_{22} & c_{23} \\ c^*_{13} & c^*_{23} & r_{33} \end{pmatrix} = I \qquad (r_{ii} : \text{real},\ c_{ij} : \text{complex}) \,. \tag{8.234}$$

Thus, there are actually three real equations and three complex equations which reduces the number of degrees of freedom by $3 + 2 \times 3 = 9$ leaving $18 - 9 = 9$ degrees of freedom for the 3×3 unitary matrix V. How many of them are associated with phases? If V is unitary, then $\{|V_{ij}|\}$ is orthogonal:

$$V_{ij} V_{kj}^* = \delta_{ik} \quad \to \quad |V_{ij}|\,|V_{kj}| = \delta_{ik}\,; \tag{8.235}$$

and a 3×3 orthogonal matrix has three degrees of freedom called the Euler angles. Thus, the absolute values of V_{ij} has 3 degrees of freedom and that leaves $9 - 3 = 6$ degrees of freedom associated with the phases θ_{ij}. Since there are only 5 degrees in the CP phases we can choose, we are actually one degree of freedom short to satisfy (8.233). Therefore, in general we *cannot* pick the CP phases such that the quark-W interaction commutes with the $\mathcal{CP}$ operator. This leads to the possibility that the quark-W coupling causes processes that are not symmetric under CP.

Incidentally, the condition (8.232) is equivalent to

$$e^{\frac{i}{2}(\phi_{u_i} - \phi_{d_j})} V_{ij} = \text{real}\,; \tag{8.236}$$

and thus it is often stated that if one can redefine the quark phases to make V_{ij} all real then CP is conserved. What really matters, however, is whether the $\mathcal{CP}$ operator commutes with the interaction Hamitonian, and there we are selecting the CP phases of the quarks.

8.5 Time reversal

In general, the symmetry under time reversal is defined in the same way as other symmetries: If one has a phonomenon that satisfies certain law of physics, then reverse the time of the phenomenon, or make a movie of it and play it backward. If the resulting phenomenon still satisfy the same law of physics, and it is the case for all phenomena that satisfy the law of physics, then that law of physics is said to be invariant under time reversal. For example, we know that the classical mechanics is invariant under time reversal. Then, if a classical motion of a point particle is given by $\vec{x}(t)$ which happens to satisfy the law of classical mechanics, then the time reversed motion $\vec{x}(-t)$ would also satisfy the law of classical mechanics. In quantum mechanics, the time-reversal invariance is stated as the invariance of transition probability when everything is time-reversed; namely, the initial and final states are interchanged where all momenta and spins are flipped while keeping particle types and energies the same:

$$|\langle i'|S|f'\rangle|^2 = |\langle f|S|i\rangle|^2\,, \tag{8.237}$$

where $|i'\rangle$ and $|f'\rangle$ represent the states $|i\rangle$ and $|f\rangle$ with momenta and spins flipped, respectively.

8.5.1 Time reversal operator and antilinearity

As in any type of symmetry in physics, the properties of the time-reversal operator can be studied when the physics is symmetric under time reversal, or at least some part of the physics is symmetric. In what follows, we will assume that the physics under consideration is invariant under time reversal. As in the case of P and C transformations, we consider an operator $\mathcal{T}$ in the Hilbert space that corresponds to time reversal that transforms particle type, 4-momenta, and spins in the expected way; namely, it flips all momenta and spins while particle types and energies are kept the same:

$$\{n_i\, \vec{p}_i\, \sigma_i\} \xrightarrow{\mathcal{T}} \{n_i\ -\vec{p}_i\ -\sigma_i\} \tag{8.238}$$

Then we will see below that such operator needs to be antilinear for self consistency, where an antilinear operator is defined by

$$\mathcal{T}(a|1\rangle + b|2\rangle) = a^*\mathcal{T}|1\rangle + b^*\mathcal{T}|2\rangle \tag{8.239}$$

where $|1\rangle$, $|2\rangle$ are any states in the Hilbert space and a, b are any complex numbers. Namely, when a complex number passes over $\mathcal{T}$, it changes to its complex conjugate. Since we are so used to linear operators, such objects may seem quite strange. We will study it in more detail later, but for now we only assume that for $\mathcal{T}$ there exists its inverse operator $\mathcal{T}^{-1}$ that satisfies

$$\mathcal{T}\mathcal{T}^{-1} = \mathcal{T}^{-1}\mathcal{T} = 1\,. \tag{8.240}$$

Suppose $|\Psi(t)\rangle$ represents a motion that satisfies the law of physics. In the Schrödinger picture, the equation of motion is

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{8.241}$$

where H is the total Hamiltonian in the Schrödinger picture, and in the interaction picture we have

$$i\frac{d}{dt}|\Psi(t)\rangle = h(t)|\Psi(t)\rangle \tag{8.242}$$

where $h(t)$ is the interaction Hamiltonian in the interaction picture. In the following we will use $h(t)$, but the argument is valid also for H. The time reversed motion would be represented by some time-dependent state which we call $|\Psi'(t)\rangle$. The situation is symbolically shown in Figure 8.4. If the physics is symmetric under time reversal, it is reasonable to assume that the state $|\Psi'(t)\rangle$ also satisfy the equation of motion:

$$i\frac{d}{dt}|\Psi'(t)\rangle = h(t)|\Psi'(t)\rangle\,. \tag{8.243}$$

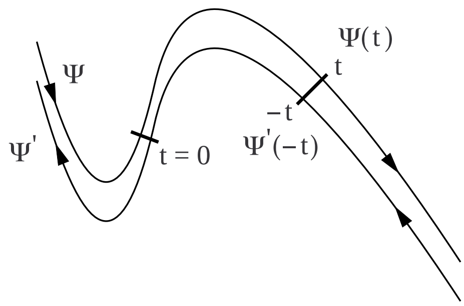


Figure 8.4: A motion of a particle represented by $\Psi(t)$ and its time-reversed motion represented by $\Psi'(t)$. If one flips the momenta and spins of the state $\Psi(t)$ at time t, it should be physically the same as $\Psi'(-t)$.

We will show that this indeed leads to the invariance of the transition probability as given by (8.237).

Along the motion $\Psi(t)$, the momenta and spins of the state in general change. Take the state $|\Psi(t)\rangle$ at time t and flip the momenta and spins, then it should be physically identical to the time-reversed state at time $-t$. Thus, we should have

$$\mathcal{T}\,|\Psi(t)\rangle = |\Psi'(-t)\rangle \tag{8.244}$$

up to a phase factor which we take to be unity for simplicity.

In the following, we will be careful not to move any complex numbers over $\mathcal{T}$ since we do not know if it is linear or antilinear at this point. Since both $|\Psi(t)\rangle$ and $|\Psi'(t)\rangle$ satisfy the same equation of motion (8.242) and (8.243), we have

$$|\Psi(t+dt)\rangle = |\Psi(t)\rangle - ih(t)\,|\Psi(t)\rangle dt\,, \tag{8.245}$$

and

$$\begin{aligned}|\Psi'(-t-dt)\rangle &= |\Psi'(-t)\rangle + ih(-t)\,|\Psi'(-t)\rangle dt \\ &= \mathcal{T}|\Psi(t)\rangle + ih(-t)\,\mathcal{T}|\Psi(t)\rangle dt\,,\end{aligned} \tag{8.246}$$

where we have used (8.244). On the other hand, we have

$$\begin{aligned}|\Psi'(-t-dt)\rangle &= \mathcal{T}|\Psi(t+dt)\rangle & \text{(by 8.244)} \\ &= \mathcal{T}\Big(|\Psi(t)\rangle - ih(t)\,|\Psi(t)\rangle dt\Big) & \text{(by 8.245)} \\ &= \mathcal{T}|\Psi(t)\rangle - \mathcal{T}ih(t)\,|\Psi(t)\rangle dt\,,\end{aligned} \tag{8.247}$$

where we took care not to move $\mathcal{T}$ over i in the last step. Comparing this with (8.246), we get

$$ih(-t)\,\mathcal{T}|\Psi(t)\rangle dt = -\mathcal{T}ih(t)\,|\Psi(t)\rangle dt\,. \tag{8.248}$$

Since this holds for any physical state $\Psi(t)$, we should have

$$ih(-t)\,\mathcal{T} = -\mathcal{T}ih(t)\,, \tag{8.249}$$

or had we used the Schrödinger picture, we would have

$$iH\,\mathcal{T} = -\mathcal{T}iH\,. \tag{8.250}$$

Now, suppose $\mathcal{T}$ is a linear operator; then this latter relation becomes

$$H\mathcal{T} = -\mathcal{T}H\,. \tag{8.251}$$

If there is no interaction, the state $|n\,\vec{p},\sigma\rangle$ satisfies

$$H|n\,\vec{p},\sigma\rangle = p^0|n\,\vec{p},\sigma\rangle \qquad (p^0 = \sqrt{\vec{p}^2+m^2})\,. \tag{8.252}$$

Applying $H\mathcal{T} = -\mathcal{T}H$ to the state $|n\,\vec{p},\sigma\rangle$, we have

$$\begin{aligned} H\mathcal{T}|n\,\vec{p},\sigma\rangle &= -\mathcal{T}H|n\,\vec{p},\sigma\rangle \\ &= -p^0\mathcal{T}|n\,\vec{p},\sigma\rangle\,; \end{aligned} \tag{8.253}$$

namely, the state $\mathcal{T}|n\,\vec{p},\sigma\rangle$ is an eigenstate of the energy with eigenvalue $-p^0 < 0$. This contradicts the assumption that $\mathcal{T}$ keeps the energy the same. Also, it is undesirable that for any state of positive energy there exists a corresponding state with negative energy. If the $\mathcal{T}$ operator is antilinear, then $iH\,\mathcal{T} = -\mathcal{T}iH$ becomes $H\mathcal{T} = \mathcal{T}H$ and the state $\mathcal{T}|n\,\vec{p},\sigma\rangle$ will have the same positive energy as $|n\,\vec{p},\sigma\rangle$.

There is other indication that the time reversal operator should be antilinear. For example, consider the commutation relation angular momentum operators

$$J_iJ_j - J_jJ_i = i\epsilon_{ijk}J_k\,, \tag{8.254}$$

where $\vec{J}$ is the angular momentum operator of the entire system under consideration where the physics is symmetric under time reversal. Since the time reversal transformation flips the sign of any angular momentum, we expect that $\vec{J}$ changes sign under time reversal:

$$\mathcal{T}\vec{J}\mathcal{T}^{-1} = -\vec{J}\,. \tag{8.255}$$

Then, applying $\mathcal{T}$ to the commutation relation above

$$\begin{aligned} &\mathcal{T}J_i \underbrace{\ }_{\mathcal{T}\mathcal{T}^{-1}} J_j\mathcal{T}^{-1} - \mathcal{T}J_j \underbrace{\ }_{\mathcal{T}\mathcal{T}^{-1}} J_i\mathcal{T}^{-1} = \mathcal{T}i\epsilon_{ijk}J_k\mathcal{T}^{-1} \\ \to \quad & J_iJ_j - J_jJ_i = \mathcal{T}i\epsilon_{ijk}J_k\mathcal{T}^{-1}\,. \end{aligned} \tag{8.256}$$

If $\mathcal{T}$ were a linear operator, i in the right hand side will simply comes out of $\mathcal{T}$, and we would have

$$J_iJ_j - J_jJ_i = i\epsilon_{ijk}\mathcal{T}J_k\mathcal{T}^{-1} = -i\epsilon_{ijk}J_k\,, \tag{8.257}$$

which is not consistent with the original commutation relation $J_iJ_j - J_jJ_i = i\epsilon_{ijk}J_k$; or, in fact, it means that $J_k = 0$ $(k = 1, 2, 3)$. If $\mathcal{T}$ were antilinear, however, the $\mathcal{T}$ transformed commutation relation above reproduces the original relation.

Hereafter, the time-reversal operator $\mathcal{T}$ in the Hilbert space is assumed to be antilinear. Then, the time-reversal condition (8.249) in the interaction picture becomes

$$\boxed{\mathcal{T}h(t)\mathcal{T}^{-1} = h(-t)}\,. \tag{8.258}$$

We have assumed that all operators in the Hilbert space are linear which can be expressed as matrices. In fact, the matrix operation on a column vector itself assumes that the operation is linear. Many features we are accustomed need to be modified because of the antilinearity, and that includes the cherished Dirac bracket notation. We will thus start from the definition of the inner product in the Hilbert space.

Antilinear operators and antiunitary operators

Suppose Ψ and Φ are arbitrary two states in the Hilbert space. The inner product of the two is a complex number denoted by (Ψ, Φ) and it satisfies

$$(\Psi, \Phi) = (\Phi, \Psi)^*\,, \tag{8.259}$$

and is linear with respect to the second argument:

$$(\Psi, a_1\Phi_1 + a_2\Phi_2) = a_1(\Psi, \Phi_1) + a_2(\Psi, \Phi_2)\,, \tag{8.260}$$

where a_1 and a_2 are any complex numbers. Taking the complex conjugate of this equation and using (8.259), we obtain

$$(a_1\Phi_1 + a_2\Phi_2, \Psi) = a_1^*(\Phi_1, \Psi) + a_2^*(\Phi_2, \Psi)\,; \tag{8.261}$$

namely, the inner product is antilinear with respect to the first argument.

A linear operator O is defined by

$$O(a_1\Psi_1 + a_2\Psi_2) = a_1O\Psi_1 + a_2O\Psi_2 \qquad (O:\ \text{linear}) \tag{8.262}$$

for any states Ψ_1 and Ψ_2. The adjoint or the hermitian conjugate of O, denoted as $O^\dagger$, is defined to be the operator that satisfies

$$(\Psi, O^\dagger\Phi) = (O\Psi, \Phi) \quad \text{for any } \Psi, \Phi\,; \tag{8.263}$$

namely, when a linear operator is moved from the first state to the second, it picks up the dagger sign. As defined earlier in (8.239), an antilinear operator A satisfies

$$A(a_1\Psi_1 + a_2\Psi_2) = a_1^* A\Psi_1 + a_2^* A\Psi_2 \qquad (A:\ \text{antilinear})\,. \tag{8.264}$$

Then, the product of an antilinear operator and a linear operator is antilinear:

$$\begin{aligned} AO(a_1\Psi_1 + a_2\Psi_2) &= A(a_1 O\Psi_1 + a_2 O\Psi_2) \\ &= a_1^* AO\Psi_1 + a_2^* AO\Psi_2\,. \end{aligned} \tag{8.265}$$

Similarly, the product of two antilinear operators is linear.

The definition of the adjoint operator (8.263) is not self-consistent for an antilinear operator as we will see below. For an antilinear operator A, suppose there exists an operator $A^\dagger$ that satisfies

$$(\Psi, A^\dagger\Phi) = (A\Psi, \Phi) \quad \text{for any } \Psi, \Phi \quad (?) \tag{8.266}$$

Now set $\Psi = i\Psi'$, then

$$\underbrace{(i\Psi', A^\dagger\Phi)}_{(8.261)\to\ -i\,(\Psi', A^\dagger\Phi)} = \underbrace{(\underbrace{A\,i}_{-iA\ \leftarrow\ (A:\ \text{antilinear})}\Psi', \Phi)}_{i(A\Psi',\Phi)\ \leftarrow\ (8.261)} \quad \to \quad -(\Psi', A^\dagger\Phi) = (A\Psi', \Phi)\,. \tag{8.267}$$

On the other hand, since (8.266) is assumed to hold for any Ψ, Φ, we have $(\Psi', A^\dagger\Phi) = (A\Psi', \Phi)$, which is inconsistent with the above unless $(\Psi', A^\dagger\Phi) = (A\Psi', \Phi) = 0$. Thus, we define the adjoint of an antilinear operator A, denoted as $A^\dagger$ using the same $\dagger$ sign as in the linear case, by

$$\boxed{(\Psi, A^\dagger\Phi) = (A\Psi, \Phi)^* \quad \text{for any } \Psi, \Phi}\,. \tag{8.268}$$

Then, it is straightforward to show that there is no inconsistency when repeating the above argument. Taking the complex conjugate of (8.268) on both sides and using (8.259), we have

$$(A^\dagger\Phi, \Psi) = (\Phi, A\Psi)^*\,; \tag{8.269}$$

namely, when one moves an antilinear operator from the first argument to the second argument of an inner product or vice versa, one has to take complex conjugate.

Using (8.268) and (8.269), we see that

$$(\Psi, (A^\dagger)^\dagger\Phi) = (A^\dagger\Psi, \Phi)^* = (\Psi, A\Phi)\,, \tag{8.270}$$

which should hold for any Ψ, Φ; thus, we have

$$\left(A^\dagger\right)^\dagger = A \quad (A:\ \text{antilinear})\,. \tag{8.271}$$

Suppose O is linear and A is antilinear. Then, the product OA is antilinear as we have seen in (8.265), and its adjoint $(OA)^\dagger$ is defined by (8.268):

$$
\begin{aligned}
(\Psi,(OA)^\dagger\Phi) &= (OA\Psi,\Phi)^* \\
(8.263)\ \rightarrow\ &= (A\Psi,O^\dagger\Phi)^* \\
(8.268)\ \rightarrow\ &= (\Psi,A^\dagger O^\dagger\Phi) \quad \text{for any } \Psi,\Phi\,.
\end{aligned}
\tag{8.272}
$$

Namely,

$$
(OA)^\dagger = A^\dagger O^\dagger\,. \tag{8.273}
$$

This can be easily extended to

$$
\boxed{(A_1A_2\cdots A_n)^\dagger = A_n^\dagger\cdots A_2^\dagger A_1^\dagger} \tag{8.274}
$$

which holds regardless of whether A_i is linear or antilinear (some may be linear and some may be antilinear).

Using the definition of adjoint for an antilinear operator (8.268) and the linearlity with respect to the second argument of inner product,

$$
\begin{aligned}
(\Psi,A^\dagger(a_1\Phi_1+a_2\Phi_2)) &= (A\Psi,a_1\Phi_1+a_2\Phi_2)^* \quad \leftarrow (8.268) \\
(8.260)\ \rightarrow\ &= a_1^*(A\Psi,\Phi_1)^*+a_2^*(A\Psi,\Phi_2)^* \\
(8.268)\ \rightarrow\ &= a_1^*(\Psi,A^\dagger\Phi_1)+a_2^*(\Psi,A^\dagger\Phi_2) \\
(8.260)\ \rightarrow\ &= (\Psi,a_1^*A^\dagger\Phi_1+a_2^*A^\dagger\Phi_2)\,.
\end{aligned}
\tag{8.275}
$$

Since this should hold for any Ψ, we have

$$
A^\dagger(a_1\Phi_1+a_2\Phi_2) = a_1^*A^\dagger\Phi_1+a_2^*A^\dagger\Phi_2\,; \tag{8.276}
$$

namely,

$$
A:\text{antilinear} \quad\rightarrow\quad A^\dagger:\text{antilinear}\,. \tag{8.277}
$$

The inverse A^{-1} of an antilinear operator A is defined as usual:

$$
A^{-1}A = AA^{-1} = 1\,. \tag{8.278}
$$

Since 1 is a linear operator, we have

$$
A:\text{antilinear} \quad\rightarrow\quad A^{-1}:\text{antilinear}\,. \tag{8.279}
$$

A unitary operator U, which is linear, may be defined by the invariance of the inner product:

$$
\boxed{(U\Psi,U\Phi) = (\Psi,\Phi) \quad \text{for any } \Psi,\Phi} \quad (\text{unitary})\,. \tag{8.280}
$$

Using the definition of the adjoint operator (8.263), we have

$$(\Psi, U^\dagger U\Phi) = (\Psi, \Phi)\,. \tag{8.281}$$

Since this should hold for any Ψ and Φ, we have $U^\dagger U = 1$. This means that $U^{-1} = U^\dagger$ and thus $UU^{-1} = UU^\dagger = 1$:

$$\boxed{U^\dagger U = UU^\dagger = 1} \quad \text{(unitary)}\,. \tag{8.282}$$

For an antilinear operator, the equation (8.280) again cannot hold for arbitrary Ψ, Φ. In fact, suppose we have

$$(V\Psi, V\Phi) = (\Psi, \Phi) \quad \text{for any } \Psi, \Phi \quad (?) \tag{8.283}$$

where V is antilinear. Then setting $\Psi = i\Psi'$ and repeating a similar argument as in (8.267), we have

$$(V\Psi', V\Phi) = -(\Psi', \Phi)\,, \tag{8.284}$$

which is inconsistent with the original relation (8.283) which should hold also for Ψ' and Φ. Instead, we define an antiunitary operator V by

$$\boxed{(V\Psi, V\Phi) = (\Psi, \Phi)^* \quad \text{for any } \Psi, \Phi} \quad \text{(antiunitary)}\,, \tag{8.285}$$

which avoids the above inconsistency as can be easily verified. Then, using the definition of adjoint for an antilinear operator (8.268),

$$(\Psi, V^\dagger V\Phi)^* = (\Psi, \Phi)^* \tag{8.286}$$

for any Ψ, Φ. Thus, we obtain $V^\dagger V = 1$. It means that $V^\dagger = V^{-1}$ and thus $VV^{-1} = VV^\dagger = 1$:

$$\boxed{V^\dagger V = VV^\dagger = 1} \quad \text{(antiunitary)}\,; \tag{8.287}$$

formally the same as in the case of an unitary operator. ▮

Exercise 8.7 *Matrix representation of an antiunitary operator.*
A vector space is spanned by a set of n basis states Ψ_i $(i = 1, \ldots, n)$. For an antilinear operator A, define a $n \times n$ matrix $a = \{a_{ij}\}$ by

$$a_{ij} \overset{\text{def}}{\equiv} (\Psi_i, A\Psi_j)\,. \tag{8.288}$$

(a) Suppose we have

$$\Phi' = A\Phi\,, \tag{8.289}$$

where the two states Φ *and* Φ' *can be expanded as*

$$\Phi = \sum_i c_i \Psi_i \,, \qquad \Phi' = \sum_i c_i' \Psi_i \,, \tag{8.290}$$

where c_i *and* c_i' *are complex numbers. Then show that the following relation holds*

$$c_i' = a_{ij} c_j^* \,. \tag{8.291}$$

(*b*) *Show that an antiunitary operator is represeted by a unitary matrix* (*preceded by complex conjugation*)*; namely, if an antilinear operator* V *satisfies* $(V\Psi, V\Phi) = (\Psi, \Phi)^*$ *for any two states* Ψ *and* Φ*, then* $v_{ij} v_{ik}^* = \delta_{jk}$ *where* $v_{ij} \equiv (\Phi_i, V\Phi_j)$*.*

As demonstrated by the above exercise, an antilinear operator may be represented by a matrix where the column vector is first taken complex conjugation then the matrix is operated on the resulting column vector. Because of this, an antilinear operator A is sometimes written as a product of a linear operator O and an 'operator' K which represents complex conjugation: $A = OK$. Such division is well defined only in terms of numerical elements for a given set of basis states, and the linear operator O corresponding to a given antilinear operator in fact depends on the choice of the basis. For example, take a 1-dimentional space spanned by a basis state Ψ and an antilinear operator defined by the identity operator I plus complex conjugation in this basis. This antilinear operator then transforms Ψ to Ψ and $i\Psi$ to $-i\Psi$. Thus, if we take $\Psi' = i\Psi$ as the basis then the corresponding linear operator is $-I$ which is different from the linear operator I in the original basis. The action of an antilinear operator, namely which state is transformed to which state, is of course independent of the choice of basis. The division of an antilinear operator into the two pieces is often a source of confusion and we will avoid using it. The matrix representation (8.291), however, does show that once the matrix elements of an antilinear operator with restpect to a set of basis states $a_{ij} \equiv (\Psi_i, A\Psi_j)$ are given, then the antilinear operator is uniquely defined. Thus, for a given matrix $\{a_{ij}\}$, we can define two different operators: one linear and the other antilinear.

We will see later that the time reversal operator $\mathcal{T}$ can be represented by a unitary matrix. Thus, the $\mathcal{T}$ operator is antiunitary according to the part (*b*) of the exercise above. Then, according to (8.287) the inverse of $\mathcal{T}$ exists which simply is its adjoint:

$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{T} = 1 \,. \tag{8.292}$$

The Dirac's bra-ket notation of a matrix element $\langle a|O|b\rangle$ assumes an associativity:

$$(\langle a|O)\,|b\rangle = \langle a|\,(O|b\rangle) \,. \tag{8.293}$$

Together with the rule that $\langle a|O$ is the adjoint of $O^\dagger|a\rangle$, this reads

$$(O^\dagger a, b) = (a, Ob) \,, \tag{8.294}$$

which is nothing but the definition of adjoint operator (8.263). Thus, the Dirac's bra-ket notation naturally assumes that the operator is linear and using the bra-ket notation to antilinear operators causes confusions when inner products are involved. Thus, we will use the explicit inner product notation (a, b) when dealing with antilinear operators.

8.5.2 S matrix and time reversal

We will now show that if the interaction Hamiltonian satisfies (8.258) or using $\mathcal{T}^{-1} = \mathcal{T}^\dagger$

$$\mathcal{T} h(t) \mathcal{T}^\dagger = h(-t) \,, \tag{8.295}$$

then the S matrix elements will indeed satisfy the condition $|\langle i'|S|f'\rangle|^2 = |\langle f|S|i\rangle|^2$ (8.237) which is the basic criteria of time invariance in quantum mechanics.

Let us recall that the S operator is the limit $t_0 \to -\infty$, $t \to \infty$ of the evolution operator $U(t, t_0)$ which satisfies

$$\Psi(t) = U(t, t_0)\Psi(t_0) \,, \tag{8.296}$$

where $\Psi(t)$ is any physical state in the interaction picture. Note that in the present notation, $\Psi(t)$ is a state in the Hilbert space and not a field operator. The U operator satisfies the equation of motion given by

$$i\frac{d}{dt}U(t, t_0) = h(t)U(t, t_0) \tag{8.297}$$

where $h(t)$ is the interaction Hamiltonian. This is valid for $t > t_0$ as well as for $t < t_0$ with the initial condition

$$U(t_0, t_0) = 1 \,. \tag{8.298}$$

Apply $\mathcal{T}$ to (8.297) to get

$$\underbrace{\mathcal{T} i}_{-i\mathcal{T}} \frac{d}{dt}U(t, t_0)\mathcal{T}^\dagger = \mathcal{T} h(t) \overbrace{\;}^{\mathcal{T}^\dagger\mathcal{T}} U(t, t_0)\mathcal{T}^\dagger \,. \tag{8.299}$$

Since dt is a real quantity, $\mathcal{T}$ and d/dt commute:

$$\begin{aligned} \mathcal{T}\frac{d}{dt}U(t, t_0) &= \mathcal{T}\frac{1}{dt}\Big[U(t + dt, t_0) - U(t, t_0)\Big] \\ &= \frac{1}{dt}\Big[\mathcal{T}U(t + dt, t_0) - \mathcal{T}U(t, t_0)\Big] = \frac{d}{dt}\Big[\mathcal{T}U(t, t_0)\Big] \,. \end{aligned} \tag{8.300}$$

Together with $\mathcal{T} h(t)\mathcal{T}^\dagger = h(-t)$, (8.299) becomes

$$-i\frac{d}{dt}\Big[\mathcal{T}U(t, t_0)\mathcal{T}^\dagger\Big] = h(-t)\Big[\mathcal{T}U(t, t_0)\mathcal{T}^\dagger\Big] \,. \tag{8.301}$$

On the other hand, setting $t = -t'$ and $t_0 = -t_0'$ in (8.297), we obtain

$$-i\frac{d}{dt}U(-t,-t_0) = h(-t)U(-t,-t_0) \tag{8.302}$$

where we have removed the primes on t and t_0. Since $\mathcal{T}U(t_0,t_0)\mathcal{T}^\dagger = \mathcal{T}\mathcal{T}^\dagger = 1$ and $U(-t_0,-t_0) = 1$, we see that $\mathcal{T}U(t,t_0)\mathcal{T}^\dagger$ and $U(-t,-t_0)$ satisfy the same initial condition and the same first-order differential equation. Thus,

$$\mathcal{T}U(t,t_0)\mathcal{T}^\dagger = U(-t,-t_0)\,. \tag{8.303}$$

Using $\Psi(t) = U(t,t_0)\Psi(t_0)$,

$$\Psi(t_0) = U(t_0,t)\,\underbrace{\Psi(t)}_{U(t,t_0)\Psi(t_0)} = U(t_0,t)U(t,t_0)\Psi(t_0) \tag{8.304}$$

which should hold for any $\Psi(t_0)$. Thus,

$$U(t_0,t)U(t,t_0) = 1 \quad \rightarrow \quad U(t_0,t) = U^{-1}(t,t_0)\,. \tag{8.305}$$

Or using the unitarity of $U(t,t_0)$,

$$U(t_0,t) = U^\dagger(t,t_0)\,. \tag{8.306}$$

Then, (8.303) becomes

$$\mathcal{T}U(t,t_0)\mathcal{T}^\dagger = U^\dagger(-t_0,-t)\,. \tag{8.307}$$

Taking the limit $t \to \infty$ and $t_0 \to -\infty$, we get

$$\mathcal{T}U(\infty,-\infty)\mathcal{T}^\dagger = U^\dagger(\infty,-\infty) \tag{8.308}$$

or using the definition $S \equiv U(\infty,-\infty)$,

$$\boxed{\mathcal{T}S\mathcal{T}^\dagger = S^\dagger \quad (T \text{ invariance})}\,. \tag{8.309}$$

Then, the S matrix element between an initial state $\Psi_i \equiv |i\rangle$ and a final $\Psi_f \equiv |f\rangle$ can be written as

$$\begin{aligned}
(\Psi_f, S\,\Psi_i) &= (\Psi_f, \mathcal{T}^\dagger\mathcal{T}\;S\;\mathcal{T}^\dagger\mathcal{T}\;\Psi_i) \\
(8.268) \rightarrow \quad &= (\mathcal{T}\Psi_f, \underbrace{\mathcal{T}S\mathcal{T}^\dagger}_{S^\dagger}\mathcal{T}\Psi_i)^* \\
(8.263) \rightarrow \quad &= (S\mathcal{T}\Psi_f, \mathcal{T}\Psi_i)^* \\
(8.259) \rightarrow \quad &= (\mathcal{T}\Psi_i, S\mathcal{T}\Psi_f)\,.
\end{aligned} \tag{8.310}$$

Thus, we have

$$\langle f|S|i\rangle = \langle i'|S|f'\rangle \tag{8.311}$$

where f' and i' are the time-reversed states of i and f, respectively:

$$|i'\rangle \equiv \Psi_{i'} \equiv \mathcal{T}\Psi_i\,, \quad |f'\rangle \equiv \Psi_{f'} \equiv \mathcal{T}\Psi_f\,. \tag{8.312}$$

Namely, the S matrix element stays the same (up to a phase) if one flips the signs of all momenta and spins and exchanges the initial and final states. This is sometimes called the reciprocity relation. It certainly satisfies the condition $|\langle i'|S|f'\rangle|^2 = |\langle f|S|i\rangle|^2$ (8.237) as promised.

Incidentally, the relation $\mathcal{T}S\mathcal{T}^\dagger = S^\dagger$ can also be derived from $\mathcal{T}h(t)\mathcal{T}^{-1} = h(-t)$ using the Dyson series which is left as an exercise. The above proof does not depend on the requirement that the S operator can be perturbatively expanded.

Exercise 8.8 *Note that the time-ordered product can be written as*

$$T(h(t_1)\cdots h(t_n)) = \sum_{(i_1\cdots i_n)} \theta(t_{i_1},\cdots,t_{i_n})h(t_{i_1})\cdots h(t_{i_n})\,, \tag{8.313}$$

where the sum is aver all possible permutation of $(1,2,\ldots,n)$ *and the real function* $\theta(t_1,\cdots,t_n)$ *is defined by*

$$\theta(t_1,\cdots,t_n) = \begin{cases} 1 & \text{if } t_1 > t_2 > \cdots > t_n \\ 0 & \text{otherwise} \end{cases}\,. \tag{8.314}$$

(*a*) *Use the* T*-invariance relation* $\mathcal{T}h(t)\mathcal{T}^{-1} = h(-t)$ *and the Hermiticity of* $h(t)$ *to show*

$$\mathcal{T}\,T(h(t_1)\cdots h(t_n))\,\mathcal{T}^\dagger = \Big(T(h(-t_n)\cdots h(-t_1))\Big)^\dagger\,. \tag{8.315}$$

(*b*) *Complete the proof of* $\mathcal{T}S\mathcal{T}^\dagger = S^\dagger$ *by directly applying* $\mathcal{T}$ *to the Dyson series*

$$S = \sum_n \frac{(-i)^n}{n!}\int dt_1\cdots dt_n T(h(t_1)\cdots h(t_n))\,.$$

8.5.3 Transformation of fields under $\mathcal{T}$

The required transformation of particle type, momentum, and spin under time reversal (8.238), indicates that the $\mathcal{T}$ operator should transform a creation operator $a^\dagger_{n\vec{p}\sigma}$ as

$$\mathcal{T}a^\dagger_{n\vec{p}\sigma}\mathcal{T}^\dagger = \zeta_{n\vec{p}\sigma}a^\dagger_{n\,-\vec{p}-\sigma}\,, \tag{8.316}$$

where $\zeta_{n\vec{p}\sigma}$ is a phase factor that may depend on particle type, momentum, and spin component along z and in general arbitrary. Under time reversal, the vacuum should transform to itself up to a phase:

$$\mathcal{T}|0\rangle = \zeta_{\rm VAC}|0\rangle\,. \tag{8.317}$$

In the case of the $\mathcal{P}$ or $\mathcal{C}$ operator, the phase assiciated with the vacuum was removed by redefining the overall phase of the operator. We could do the same here by defining $\mathcal{T} \equiv \zeta_{\rm VAC}\mathcal{T}'$:

$$\underbrace{\mathcal{T}}_{\zeta_{\rm VAC}\mathcal{T}'}|0\rangle = \zeta_{\rm VAC}|0\rangle \quad \rightarrow \quad \mathcal{T}'|0\rangle = |0\rangle\,. \tag{8.318}$$

Due to the antilinearlity of the $\mathcal{T}$ operator, however, we could remove the phase also by simply redefining the phase of the vacuum state. In general, if $\mathcal{T}$ transforms a state to itself up to a phase

$$\mathcal{T}|\Psi\rangle = e^{i\phi}|\Psi\rangle\,, \tag{8.319}$$

then by defining $|\psi\rangle \equiv e^{-i\frac{\phi}{2}}|\Psi'\rangle$, we have

$$\underbrace{\mathcal{T}e^{-i\frac{\phi}{2}}}_{e^{i\frac{\phi}{2}}\mathcal{T}}|\Psi'\rangle = e^{i\phi}e^{-i\frac{\phi}{2}}|\Psi'\rangle \quad \rightarrow \quad \mathcal{T}|\Psi'\rangle = |\Psi'\rangle\,. \tag{8.320}$$

Thus, the phase factor as an eigenvalue of $\mathcal{T}$ has no physical meaning. Hereafter, we assume that the time-reversal phase of the vacuum is $+1$:

$$\mathcal{T}|0\rangle = |0\rangle\,. \tag{8.321}$$

Then, (8.316) defines the transformation of the basis states of the Hilbert space:

$$\begin{aligned}
\mathcal{T}|\{n_i\,\vec{p}_i\,\sigma_i\}\rangle &= \mathcal{T}a^\dagger_{n_1\,\vec{p}_1,\sigma_1}\underbrace{\cdots}_{\mathcal{T}^\dagger\mathcal{T}}\cdots\underbrace{\cdots}_{\mathcal{T}^\dagger\mathcal{T}}a^\dagger_{n_k\,\vec{p}_k,\sigma_k}\underbrace{\ }_{\mathcal{T}^\dagger\mathcal{T}}|0\rangle \\
&= \zeta_{n_1\,\vec{p}_1,\sigma_1}\cdots\zeta_{n_k\,\vec{p}_k,\sigma_k}a^\dagger_{n_1\,-\vec{p}_1,-\sigma_1}\cdots a^\dagger_{n_k\,-\vec{p}_k,-\sigma_k}|0\rangle \\
&= \zeta_{n_1\,\vec{p}_1,\sigma_1}\cdots\zeta_{n_k\,\vec{p}_k,\sigma_k}|\{n_i\,-\vec{p}_i\,-\sigma_i\}\rangle \\
&= \zeta_{\{n_i\,\vec{p}_i,\sigma_i\}}|\{n_i\,-\vec{p}_i\,-\sigma_i\}\rangle\,,
\end{aligned} \tag{8.322}$$

where we have defined

$$\zeta_{\{n_i\,\vec{p}_i,\sigma_i\}} \equiv \zeta_{n_1\,\vec{p}_1,\sigma_1}\cdots\zeta_{n_k\,\vec{p}_k,\sigma_k}\,. \tag{8.323}$$

The matrix elements of $\mathcal{T}$ with respect to the basis states are then given by

$$(\Psi_{\{n_i\,\vec{p}_i\,\sigma_i\}}, \mathcal{T}\Psi_{\{n_j\,\vec{p}_j\,\sigma_j\}}) = \zeta_{\{n_i\,\vec{p}_i,\sigma_i\}}\delta_{\{n_i\,\vec{p}_i,\sigma_i\}\{n_j\,-\vec{p}_j,-\sigma_j\}} \tag{8.324}$$

where we have used the notation $\Psi_{\{n_i\,\vec{p}_i\,\sigma_i\}} \equiv |\{n_i\,\vec{p}_i\,\sigma_i\}\rangle$ and $\delta_{\{n_i\,\vec{p}_i,\sigma_i\}\{n_j\,-\vec{p}_j,-\sigma_j\}}$ is defined to be $+1$ only if the particle types are all the same and momenta and spins are relatively opposite sign between the two groups, and otherwise the value is zero. In the huge matrix formed by the above matrix elements, there is only one non-zero element in each row or in each column which is a phase factor. This matrix is thus a unitary matrix, and $\mathcal{T}$, which is an antilinear operator represented by a unitary matrix, is then antiunitary.

Now we will attempt to adjust the arbitrary time-reversal phases $\zeta_{n\,\vec{p}\,\sigma}$ such that the Yukawa interaction Hamiltonian $h_Y(t) = \int d^3x\,\bar{\psi}\psi\,\phi$ and the QED hamiltonian $h_{QED}(t) = \int d^3x\,\bar{\psi}\gamma_\mu\psi\,A^\mu$ satisfies the condition of time-reversal invariance $\mathcal{T}h(t)\mathcal{T}^\dagger = h(-t)$. If we can choose such phases, then the corresponding physics is invariant under time reversal.

For a spin-0 particle we drop the spin index and thus (8.316) reads

$$\mathcal{T}a^\dagger_{s\,\vec{p}}\mathcal{T}^\dagger = \zeta_{s\,\vec{p}}\,a^\dagger_{s\,-\vec{p}} \xrightarrow{\text{take}\dagger} \mathcal{T}a_{s\,\vec{p}}\mathcal{T}^\dagger = \zeta^*_{s\,\vec{p}}\,a_{s\,-\vec{p}}\,. \tag{8.325}$$

Applying $\mathcal{T}$ to the charged spin-0 field (8.41),

$$\begin{aligned}
\mathcal{T}\phi(x)\mathcal{T}^\dagger &= \mathcal{T}\sum_{\vec{p}}\Big(a_{s\,\vec{p}}\,e_{\vec{p}}(x) + a^\dagger_{\bar{s}\,\vec{p}}\,e^*_{\vec{p}}(x)\Big)\,\mathcal{T}^\dagger \\
&= \sum_{\vec{p}}\Big((\mathcal{T}a_{s\,\vec{p}}\mathcal{T}^\dagger)\,\underset{\uparrow}{e^*_{\vec{p}}(x)} + (\mathcal{T}a^\dagger_{\bar{s}\,\vec{p}}\mathcal{T}^\dagger)\,\underset{\uparrow}{e_{\vec{p}}(x)}\Big) \\
&\qquad\qquad \text{conjugated since } \mathcal{T}^\dagger \text{ moved over them} \\
&= \sum_{\vec{p}}\Big(\zeta^*_{s\,\vec{p}}\,a_{s\,-\vec{p}}\,e^*_{\vec{p}}(x) + \zeta_{\bar{s}\,\vec{p}}\,a^\dagger_{\bar{s}\,-\vec{p}}\,e_{\vec{p}}(x)\Big) \\
(\vec{p}\to-\vec{p}) \quad &= \sum_{\vec{p}}\Big(\zeta^*_{s\,-\vec{p}}\,a_{s\,\vec{p}}\,e^*_{-\vec{p}}(x) + \zeta_{\bar{s}\,-\vec{p}}\,a^\dagger_{\bar{s}\,\vec{p}}\,e_{-\vec{p}}(x)\Big) \\
(x'\equiv Tx) \quad &= \sum_{\vec{p}}\Big(\zeta^*_{s\,-\vec{p}}\,a_{s\,\vec{p}}\,e_{\vec{p}}(x') + \zeta_{\bar{s}\,-\vec{p}}\,a^\dagger_{\bar{s}\,\vec{p}}\,e^*_{\vec{p}}(x')\Big)
\end{aligned} \tag{8.326}$$

where we have used $x' = Tx = (-x^0, \vec{x})$ and thus

$$e_{-\vec{p}}(x) = \frac{e^{-ip^0x^0 - i\vec{p}\cdot\vec{x}}}{\sqrt{2p^0V}} = \frac{e^{ip^0x'^0 - i\vec{p}\cdot\vec{x}'}}{\sqrt{2p^0V}} = e^*_{\vec{p}}(x')\,. \tag{8.327}$$

As in the case of the transformation under $\mathcal{P}$ or $\mathcal{C}$, we require that the transformed field should be somehow ‘proportional’ to the original field. then, the phase factors should not depend on $\vec{p}$: $\zeta_{s\,\vec{p}} = \zeta_s$, $\zeta_{\bar{s}\,\vec{p}} = \zeta_{\bar{s}}$, and

$$\zeta_{\bar{s}} = \zeta^*_s\,. \tag{8.328}$$

Then, we have

$$\mathcal{T}\phi(x)\mathcal{T}^\dagger = \zeta^*_s\phi(x') \qquad (x'\equiv Tx)\,. \tag{8.329}$$

We will see later that this is not an over-constraint on the choice of the time-reversal phases at least for the Yukawa coupling and the QED coupling; namely, we can adjust the particle specific phases only and make them satisfy $\mathcal{T}h(t)\mathcal{T}^\dagger = h(-t)$. In fact, if the interaction cannot be made to satisfy the condition of time-reversal invariance by choosing the particle specific phase ζ_s only, then in general it cannot be done if the phase are chosen to be different for partcles and antiparticles or for different momenta.

Moving on to the fermion field, the transformation (8.316) gives

$$\mathcal{T} a^\dagger_{f\vec{p}\sigma}\mathcal{T}^\dagger = \zeta_{f\vec{p}\sigma}\, a^\dagger_{f-\vec{p}-\sigma} \quad \xrightarrow{\text{take}\dagger} \quad \mathcal{T} a_{f\vec{p}\sigma}\mathcal{T}^\dagger = \zeta^*_{f\vec{p}\sigma}\, a_{f-\vec{p}-\sigma}\,. \tag{8.330}$$

Applying $\mathcal{T}$ to the momentum expansion (8.43) and taking the complex conjugates of $f_{\vec{p}\sigma}$ and $g_{\vec{p}\sigma}$ as we pass $\mathcal{T}^\dagger$ over them,

$$\begin{aligned}
\mathcal{T}\psi(x)\mathcal{T}^\dagger &= \sum_{\vec{p},\sigma}\Big(\mathcal{T} a_{f\vec{p}\sigma}\mathcal{T}^\dagger\, f^*_{\vec{p}\sigma}(x) + \mathcal{T} a^\dagger_{\bar{f}\vec{p}\sigma}\mathcal{T}^\dagger g^*_{\vec{p}\sigma}(x)\Big) \\
&= \sum_{\vec{p},\sigma}\Big(\zeta^*_{f\vec{p}\sigma}\, a_{f-\vec{p}-\sigma} f^*_{\vec{p}\sigma}(x) + \zeta_{\bar{f}\vec{p}\sigma}\, a^\dagger_{\bar{f}-\vec{p}-\sigma} g^*_{\vec{p}\sigma}(x)\Big) \\
\begin{pmatrix} \vec{p}\to -\vec{p} \\ \sigma \to -\sigma \end{pmatrix} \quad &= \sum_{\vec{p},\sigma}\Big(\zeta^*_{f-\vec{p}-\sigma}\, a_{f\vec{p}\sigma} f^*_{-\vec{p}-\sigma}(x) + \zeta_{\bar{f}-\vec{p}-\sigma}\, a^\dagger_{\bar{f}\vec{p}\sigma} g^*_{-\vec{p}-\sigma}(x)\Big)\,.
\end{aligned} \tag{8.331}$$

Using $e^*_{-\vec{p}}(x) = e_{\vec{p}}(x')$ with $x' = Tx = (-x^0, \vec{x})$ (8.327),

$$\begin{aligned}
f^*_{-\vec{p}-\sigma}(x) &= u^*_{-\vec{p}-\sigma}\, e^*_{-\vec{p}}(x) = u^*_{-\vec{p}-\sigma}\, e_{\vec{p}}(x') \\
g^*_{-\vec{p}-\sigma}(x) &= v^*_{\vec{p}-\sigma}\, e_{-\vec{p}}(x) = v^*_{-\vec{p}-\sigma}\, e^*_{\vec{p}}(x')\,.
\end{aligned} \tag{8.332}$$

Thus, $\mathcal{T}\psi(x)\mathcal{T}^\dagger$ is now

$$\mathcal{T}\psi(x)\mathcal{T}^\dagger = \sum_{\vec{p},\sigma}\Big(\zeta^*_{f-\vec{p}-\sigma}\, a_{f\vec{p}\sigma} u^*_{-\vec{p}-\sigma}\, e_{\vec{p}}(x') + \zeta_{\bar{f}-\vec{p}-\sigma}\, a^\dagger_{\bar{f}\vec{p}\sigma} v^*_{-\vec{p}-\sigma}\, e^*_{\vec{p}}(x')\Big)\,. \tag{8.333}$$

As in the spin-0 case, we would like to express the transformed field in terms of original field. To do so, we have to somehow realte $u^*_{-\vec{p}-\sigma}$ and $v^*_{-\vec{p}-\sigma}$ to $u_{\vec{p}\sigma}$ and $v_{\vec{p}\sigma}$, respectively. We will now show that the following relations hold

$$u^*_{-\vec{p}-\sigma} = s_\sigma\, \delta\, \mathrm{T} u_{\vec{p}\sigma}\,, \quad v^*_{-\vec{p}-\sigma} = s_\sigma\, \bar{\delta}\, \mathrm{T} v_{\vec{p}\sigma} \qquad (s_\sigma \overset{\text{def}}{\equiv} \mathrm{sign}(\sigma))\,, \tag{8.334}$$

where δ and $\bar{\delta}$ are some phase factors that do not depend on σ, and the 4×4 matrix T is unitary and anti-symmetric

$$\boxed{\mathrm{T}^\dagger\mathrm{T} = 1\,, \quad \mathrm{T}^T = -\mathrm{T}}\,, \tag{8.335}$$

and satisfies

$$\boxed{\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma^*_\mu \quad (\mu = 0,1,2,3)}\,. \tag{8.336}$$

Note that the index μ changed from a superscript to a subscript. Also, T is a 4×4 matrix that operates in the spinor space (in particular it is not antilinear) and we are distinguishing it from the antilinear operator $\mathcal{T}$ which operates in the Hilbert space and from the Lorentz transformation matrix T which operates in the space-time four-dimension.

Such a T exists in the Dirac representation:

$$\mathrm{T} = \gamma_3\gamma_1 = -i\Sigma_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{(Dirac rep.)}\,, \tag{8.337}$$

which is clearly anti-symmetric. Using $\gamma_\mu^\dagger = \gamma^\mu$,

$$(\gamma_3\gamma_1)^\dagger(\gamma_3\gamma_1) = \underbrace{\gamma_1^\dagger\gamma_3^\dagger}_{\gamma^1\gamma^3}\gamma_3\gamma_1 = 1\,, \tag{8.338}$$

namely T is unitary. Since in the Dirac representation γ_μ's are all real except for γ_2 which is purely imaginary, (8.336) indicates that T commutes with γ_0 and γ_2 and anticommutes with γ_1 and γ_3. Thus, $\mathrm{T} = \gamma_3\gamma_1$ fits the bill. In a different representation transformed by a unitary matrix V ($\gamma'_\mu = V\gamma_\mu V^\dagger$), the appropriate T' is given by

$$\mathrm{T}' = V^*\mathrm{T}V^\dagger\,, \tag{8.339}$$

as it is straightforward to show that it is unitary, anti-symmetric, and $\mathrm{T}'\gamma^{\mu\prime}\mathrm{T}'^\dagger = \gamma'^*_\mu$.

Exercise 8.9 *Representation independence of* T.
Suppose a 4×4 *matrix* T *is unitary and antisymmetric and satisfies* $\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma^*_\mu$ ($\mu = 0,1,2,3$). *When one changes the representation of* γ_μ *by* $\gamma'_\mu = V\gamma_\mu V^\dagger$, *show that the matrix given by* $\mathrm{T}' \equiv V^*TV^\dagger$ *is also unitary and antisymmetic and satisfies* $\mathrm{T}'\gamma'^\mu\mathrm{T}'^\dagger = \gamma'^*_\mu$ ($\mu = 0,1,2,3$).

Now recall that the u, v spinors with general $\vec{p}$ are obtained by

$$u_{\vec{p}\sigma} \equiv e^{\frac{1}{2}\xi_i\alpha_i}u_{\vec{0}\sigma}\,,\quad v_{\vec{p}\sigma} \equiv e^{\frac{1}{2}\xi_i\alpha_i}v_{\vec{0}\sigma} \tag{8.340}$$

where $u_{\vec{0}\sigma}$ and $v_{\vec{0}\sigma}$ are the electron and positron spinors at rest with spin component along z given by σ, and $\vec{\xi}$ is the vector in the direction of the boost with $\tan|\vec{\xi}| = |\vec{p}|/p^0$. First, we will show that

$$u^*_{\vec{0}-\frac{1}{2}} = \delta\,\mathrm{T}u_{\vec{0}+\frac{1}{2}}\,, \tag{8.341}$$

where δ is some phase factor that depends on the phase convention of the two spinors. As in the case of charge conjugation, the strategy is to recall that for any momentum $\vec{p}$ and spin quantization direction $\vec{s}$, the 4 eigen spinors of $\not{p}/m$ and $\gamma_5\not{s}$ form a complete orthogonal basis, and with proper normalization and phase convention, they are ($u_{\vec{p}\sigma}$, $v_{\vec{p}\sigma}$ $\sigma = \pm 1/2$). Thus, if two spinors have the same eigenvalues for $\not{p}/m$ and $\gamma_5 \not{s}$, then they should be the same up to a constant multiplicative factor. In our case at hand, we have $\vec{p} = 0$ and $\vec{s} = \hat{z}$; thus,

$$\frac{\not{p}}{m} = \gamma^0 \,, \quad \text{and} \quad \gamma_5 \not{s} = -\gamma_5 \gamma^3 = -i\gamma^0\gamma^1\gamma^2 \underbrace{\gamma^3\gamma^3}_{-1} = \gamma^0 \Sigma_3 \,, \tag{8.342}$$

where we have used $\Sigma_3 = i\gamma^1\gamma^2$. Thus, the two matrices are γ^0 and $\gamma^0\Sigma_3$, or equivalently, γ^0 and Σ_3. Since $u_{\vec{0}+\frac{1}{2}}$ is an eigenspinor of $\not{p}/m = \gamma^0$ with eigenvalue $+1$,

$$\overbrace{\gamma^0}^{\mathrm{T}} \overbrace{}^{\mathrm{T}^\dagger \mathrm{T}} u_{\vec{0}+\frac{1}{2}} = \overbrace{u_{\vec{0}+\frac{1}{2}}}^{\mathrm{T}} \quad \to \quad \underbrace{\mathrm{T}\gamma^0\mathrm{T}^\dagger}_{\gamma_0^* \text{ by } (8.336)} \mathrm{T}u_{\vec{0}+\frac{1}{2}} = \mathrm{T}u_{\vec{0}+\frac{1}{2}} \,. \tag{8.343}$$

Taking the complex conjugate, we obtain

$$\gamma^0 (\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* = (\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* \,. \tag{8.344}$$

On the other hand, $u_{\vec{0}+\frac{1}{2}}$ is an eigenspinor of $\Sigma_3 = i\gamma^1\gamma^2$ with eigenvalue $+1$:

$$\overbrace{i\gamma^1}^{\mathrm{T}} \overbrace{}^{\mathrm{T}^\dagger \mathrm{T}} \overbrace{\gamma^2}^{} \overbrace{}^{\mathrm{T}^\dagger \mathrm{T}} u_{\vec{0}+\frac{1}{2}} = \overbrace{u_{\vec{0}+\frac{1}{2}}}^{\mathrm{T}} \quad \to \quad i \underbrace{\mathrm{T}\gamma^1\mathrm{T}^\dagger\mathrm{T}\gamma^2\mathrm{T}^\dagger}_{\gamma_1^*\gamma_2^* \text{ by } (8.336)} \mathrm{T}u_{\vec{0}+\frac{1}{2}} = \mathrm{T}u_{\vec{0}+\frac{1}{2}} \,. \tag{8.345}$$

Taking the complex conjugate of this,

$$-i\gamma^1\gamma^2 (\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* = (\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* \quad \to \quad \Sigma_3 (\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* = -(\mathrm{T}u_{\vec{0}+\frac{1}{2}})^* \,. \tag{8.346}$$

Namely, $(\mathrm{T}u_{\vec{0}+\frac{1}{2}})^*$ and $u_{\vec{0}-\frac{1}{2}}$ has the same eigenvalues for $\not{p}/m = \gamma^0$ and Σ_3. Also, it is properly normalized due to the unitarity of T; thus, they are the same up to a phase and this proves $u^*_{\vec{0}-1/2} = \delta\, \mathrm{T}u_{\vec{0}+1/2}$ (8.341). Now taking its complex conjugate and left-multiplying with T, we obtain

$$\mathrm{T}u_{\vec{0}-\frac{1}{2}} = \delta^* \underbrace{\mathrm{T}\mathrm{T}^*}_{-1} u^*_{\vec{0}+\frac{1}{2}} \quad \to \quad -u^*_{\vec{0}+\frac{1}{2}} = \delta\, \mathrm{T}u_{\vec{0}-\frac{1}{2}} \,, \tag{8.347}$$

where we have used

$$\mathrm{T}^\dagger\mathrm{T} = 1 \quad \xrightarrow{T} \quad \underbrace{\mathrm{T}^T}_{-\mathrm{T}} \mathrm{T}^* = 1 \quad \to \quad \mathrm{T}\mathrm{T}^* = -1 \,. \tag{8.348}$$

The relations (8.347) and (8.341) can be summarized as

$$u^*_{\vec{0}-\sigma} = s_\sigma\, \delta\, \mathrm{T}u_{\vec{0}\sigma} \,. \tag{8.349}$$

The transformation of the boost matrix $e^{\frac{1}{2}\xi_i\alpha_i}$ by T is easily found by (8.336): using $\alpha_i = \gamma^0\gamma^i$,

$$\mathrm{T}\alpha_i\mathrm{T}^\dagger = \mathrm{T}\gamma^0\gamma^i\mathrm{T}^\dagger = \gamma_0^*\gamma_i^* = -(\gamma^0\gamma^i)^* = -\alpha_i^* \,; \tag{8.350}$$

thus,

$$\mathrm{T}e^{\frac{1}{2}\xi_i\alpha_i}\mathrm{T}^\dagger = (e^{-\frac{1}{2}\xi_i\alpha_i})^* \quad \to \quad \mathrm{T}e^{\frac{1}{2}\xi_i\alpha_i} = (e^{-\frac{1}{2}\xi_i\alpha_i})^*\mathrm{T}\,. \tag{8.351}$$

Applying $(e^{-\frac{1}{2}\xi_i\alpha_i})^*$ to (8.349),

$$\underbrace{(e^{-\frac{1}{2}\xi_i\alpha_i}u_{\vec{0}\,-\sigma})^*}_{u^*_{-\vec{p}\,-\sigma}} = s_\sigma\,\delta\,\underbrace{(e^{-\frac{1}{2}\xi_i\alpha_i})^*\mathrm{T}}_{\mathrm{T}e^{\frac{1}{2}\xi_i\alpha_i}}\,u_{\vec{0}\,\sigma} = s_\sigma\,\delta\,\mathrm{T}u_{\vec{p}\sigma}\,, \tag{8.352}$$

which proves the first of (8.334). The second relation which is for v spinors is obtained by repeating the same procedure for v spinors. The two phases δ and $\bar{\delta}$ are in general different, and T cannot absorb both phases.

Exercise 8.10 *Follow the same argument as above and prove* $v^*_{-\vec{p}\,-\sigma} = s_\sigma\,\bar{\delta}\,\mathrm{T}v_{\vec{p}\sigma}$.

We can now use (8.334) in the time reversal of fermion field (8.333):

$$\begin{aligned}\mathcal{T}\psi(x)\mathcal{T}^\dagger &= \sum_{\vec{p},\sigma}\Big(\zeta^*_{f-\vec{p}-\sigma}\,a_{f\vec{p}\sigma}s_\sigma\,\delta\,\mathrm{T}u_{\vec{p}\sigma}\,e_{\vec{p}}(x') + \zeta_{\bar{f}-\vec{p}-\sigma}\,a^\dagger_{\bar{f}\vec{p}\sigma}s_\sigma\,\bar{\delta}\,\mathrm{T}v_{\vec{p}\sigma}\,e^*_{\vec{p}}(x')\Big)\\ &= \mathrm{T}\sum_{\vec{p},\sigma}\Big(\zeta^*_{f-\vec{p}-\sigma}s_\sigma\,\delta\,a_{f\vec{p}\sigma}f_{\vec{p}\sigma}(x') + \zeta_{\bar{f}-\vec{p}-\sigma}s_\sigma\,\bar{\delta}\,a^\dagger_{\bar{f}\vec{p}\sigma}g_{\vec{p}\sigma}(x')\Big)\,. \end{aligned}\tag{8.353}$$

In order for this to be 'proportional' to $\psi(x')$, the phase factors in front of $a_{f\vec{p}\sigma}$ and $a^\dagger_{\bar{f}\vec{p}\sigma}$ should be independent of $\vec{p}$, σ, and particle or antiparticle:

$$\zeta^*_{f-\vec{p}-\sigma}s_\sigma\,\delta = \zeta_{\bar{f}-\vec{p}-\sigma}s_\sigma\,\bar{\delta} \overset{\text{def}}{\equiv} \zeta^*_f\,. \tag{8.354}$$

Then, we have

$$\mathcal{T}\psi(x)\mathcal{T}^\dagger = \zeta^*_f\mathrm{T}\psi(x') \qquad (x' \equiv Tx)\,. \tag{8.355}$$

Incidentally, the condition (8.354) means that the time-reversal phases for fermions can be written as

$$\zeta_{f\vec{p}\sigma} = \delta\,s_{-\sigma}\,\zeta_f\,, \quad \zeta_{\bar{f}\vec{p}\sigma} = \bar{\delta}^*\,s_{-\sigma}\,\zeta^*_f\,. \tag{8.356}$$

The factor $s_{-\sigma}$ causes a relative sign difference between $\sigma = +1/2$ and $-1/2$ for given particle type and momentum. Note that this is entirely independent of the phase convention used for the u and v spinors since the effect of the phase convention is explicitly contained in the phase factors δ and $\bar{\delta}$. Then where did the relative minus sign come from? It came from $TT^* = -1$ used in (8.347) which in turn was due to the antisymmetry of T. Then, a natural question is whether T had to antisymmetric.

At least the solution $\mathrm{T} = \gamma^1\gamma^3$ (8.337) in the Dirac representation is antisymmetric. As shown in Problem 8.3, one can use the Pauli's fundamental theorem to show that the unitary matrix that satisfies $\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma^*_\mu$ is unique up to a phase factor. Thus, T has to be asymmetric in the Dirac representation, and thus in any representation according to Exercise 8.9.

We proceed similarly for a charged spin-1 field. The transformation of creation and annihilation operators are

$$\mathcal{T} a^\dagger_{v\,\vec{p}\sigma}\mathcal{T}^\dagger = \zeta_{v\,\vec{p}\sigma}\, a^\dagger_{v\,-\vec{p}-\sigma} \xrightarrow{\text{take}\dagger} \mathcal{T} a_{v\,\vec{p}\sigma}\mathcal{T}^\dagger = \zeta^*_{v\,\vec{p}\sigma}\, a_{v\,-\vec{p}-\sigma}\,. \tag{8.357}$$

Then, the momentum expansion of A^μ transforms as

$$\begin{aligned} \mathcal{T} A^\mu(x)\mathcal{T}^\dagger &= \sum_{\vec{p},\sigma} \Big(\underbrace{\mathrm{T} a_{v\,\vec{p}\sigma}\mathrm{T}^\dagger}_{\zeta^*_{v\,\vec{p}\sigma}a_{v\,-\vec{p}-\sigma}} \epsilon^{\mu*}_{\vec{p}\sigma} e^*_{\vec{p}}(x) + \underbrace{\mathrm{T} a^\dagger_{\bar{v}\,\vec{p}\sigma}\mathrm{T}^\dagger}_{\zeta_{\bar{v}\,\vec{p}\sigma}a^\dagger_{\bar{v}\,-\vec{p}-\sigma}} \epsilon^{\mu}_{\vec{p}\sigma} e_{\vec{p}}(x)\Big) \\ \begin{pmatrix} \vec{p}\to-\vec{p} \\ \sigma\to-\sigma \end{pmatrix} &= \sum_{\vec{p},\sigma} \Big(\zeta^*_{v\,-\vec{p}-\sigma} a_{v\,\vec{p}\sigma}\epsilon^{\mu*}_{-\vec{p}-\sigma} e^*_{-\vec{p}}(x) + \zeta_{\bar{v}\,-\vec{p}-\sigma} a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^{\mu}_{-\vec{p}-\sigma} e_{-\vec{p}}(x)\Big)\,. \end{aligned} \tag{8.358}$$

Using the definition $\epsilon_{\vec{p}\sigma} \equiv \Lambda_{\vec{p}}\,\epsilon_{\vec{0}\,\sigma}$ and the relation

$$\epsilon^*_{\vec{0}\,-\sigma} = (-)^\sigma \epsilon_{\vec{0}\,\sigma} \tag{8.359}$$

which can be obtained from the explicit expression (5.209), we have

$$\epsilon^*_{-\vec{p},-\sigma} = \underbrace{\Lambda_{-\vec{p}}}_{(8.58)\,\to\, P\Lambda_{\vec{p}}P} \underbrace{\epsilon^*_{\vec{0},-\sigma}}_{(-)^\sigma\epsilon_{\vec{0}\,\sigma}} = (-)^\sigma P\Lambda_{\vec{p}} \underbrace{P\epsilon_{\vec{0}\,\sigma}}_{-\epsilon_{\vec{0}\,\sigma}\ (\text{time component}\,=\,0)} = (-)^{1+\sigma} P\epsilon_{\vec{p}\sigma}\,, \tag{8.360}$$

or

$$\epsilon^{\mu*}_{-\vec{p}-\sigma} = (-)^{1+\sigma}\epsilon_{\vec{p}\sigma_\mu}\,. \tag{8.361}$$

Together with $e^*_{-\vec{p}}(x) = e_{\vec{p}}(x')$ $(x' = Tx)$, we then have

$$\begin{aligned} \mathcal{T} A^\mu(x)\mathcal{T}^\dagger = \sum_{\vec{p},\sigma} \Big(&\zeta^*_{v\,-\vec{p}-\sigma}(-)^{1+\sigma} a_{v\,\vec{p}\sigma}\epsilon_{\vec{p}\sigma_\mu} e_{\vec{p}}(x') \\ &+ \zeta_{\bar{v}\,-\vec{p}-\sigma}(-)^{1+\sigma} a^\dagger_{\bar{v}\,\vec{p}\sigma}\epsilon^*_{\vec{p}\sigma_\mu} e^*_{\vec{p}}(x')\Big)\,. \end{aligned} \tag{8.362}$$

Again in order for this be 'proportional' to $A^\mu(x')$, the phase factors should be independent of particle type, momentum, and spin:

$$\zeta^*_{v\,-\vec{p}-\sigma}(-)^{1+\sigma} = \zeta_{\bar{v}\,-\vec{p}-\sigma}(-)^{1+\sigma} \stackrel{\text{def}}{\equiv} \zeta^*_v\,. \tag{8.363}$$

Then, the transformation of a spin-1 field becomes

$$\mathcal{T} A^\mu(x)\mathcal{T}^\dagger = \zeta^*_v A_\mu(x') \qquad (x' \equiv Tx)\,. \tag{8.364}$$

The phase condition (8.363) can be written as

$$\zeta_{v\,\vec{p}\sigma} = (-)^{1-\sigma}\zeta_v\,, \quad \zeta_{\bar{v}\,\vec{p}\sigma} = (-)^{1-\sigma}\zeta_v^*\,. \tag{8.365}$$

Again, the time-reversal phase alternates the sign as one steps through σ as in the case of the fermion field. Is this independent of the phase convention used for the normal mode function? Unlike the fermion case, this is actually phase-convention dependent. We have defined the polarization vectors to satisfy $\epsilon^*_{\vec{0}\,-1} \equiv -\epsilon^*_{\vec{0}\,+1}$, but we could have used $\epsilon^*_{\vec{0}\,-1} \equiv \epsilon^*_{\vec{0}\,+1}$ which would give $\epsilon^*_{\vec{0}\,-\sigma} = \epsilon_{\vec{0}\sigma}$ instead of (8.359). This would result in no σ-dependence of the time reversal phases for a spin-1 particle.

We have so far avoided as much as possible to use specific phase conventions in order to elucidate the essential logic. It is, however, useful sometimes to adopt some phase conventions that is consistent with the standard phase conventions of angular momentum. In particular, it is customary to fix the relative phases of different σ by the raising and lowering operators $J_\pm = J_x \pm iJ_y$. Namely for states at rest, we define

$$J_\pm|n\,\vec{0}\,\sigma\rangle = \sqrt{(j\mp\sigma)(j\pm\sigma+1)}\,|n\,\vec{0}\,\sigma{\pm}1\rangle\,, \tag{8.366}$$

Applying $\mathcal{T}$ to this,

$$\mathcal{T}J_\pm|n\,\vec{0}\,\sigma\rangle = \sqrt{(j\mp\sigma)(j\pm\sigma+1)}\,\underbrace{\mathcal{T}|n\,\vec{0}\,\sigma{\pm}1\rangle}_{\zeta_{n\,\vec{0}\,\sigma\pm1}|n\,\vec{0}\;-(\sigma\pm1)\rangle}\,. \tag{8.367}$$

Using the antilinearity of $\mathcal{T}$ and $\mathcal{T}J_i\mathcal{T}^\dagger = -J_i$ (8.255) or $\mathcal{T}J_i = -J_i\mathcal{T}$,

$$\mathcal{T}J_\pm = \mathcal{T}J_1 \pm \underbrace{\mathcal{T}\,i}_{-i\mathcal{T}}\,J_2 = -J_1\mathcal{T} \pm iJ_2\mathcal{T} = -J_\mp\mathcal{T}\,. \tag{8.368}$$

Thus,

$$\begin{aligned}\mathcal{T}J_\pm|n\,\vec{0}\,\sigma\rangle &= -J_\mp\underbrace{\mathcal{T}|n\,\vec{0}\,\sigma\rangle}_{\zeta_{n\,\vec{o}\,\sigma}|n\,\vec{0}\,-\sigma\rangle} = -\zeta_{n\,\vec{0}\,\sigma}J_\mp|n\,\vec{0}\;-\sigma\rangle \\ &= -\zeta_{n\,\vec{0}\,\sigma}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}|n\,\vec{0}\;-(\sigma\pm1)\rangle\,. \end{aligned}\tag{8.369}$$

Comparing this and (8.367), we have

$$\zeta_{n\,\vec{0}\,\sigma\pm1} = -\zeta_{n\,\vec{0}\,\sigma}\,. \tag{8.370}$$

Since in general the time-reversal phase should not depend on momentum, we have $\zeta_{n\,\vec{p}\sigma\pm1} = -\zeta_{n\,\vec{p}\sigma}$ also for non-zero $\vec{p}$. Thus, if we adopt the standard phase convention for different values of spin z-component σ, the time reversal phase should alternate the sign as σ is varied from $-j$ to $+j$ where j is the absolute spin of the particle. The dependence on j is a dependence on the particle type; thus, we can write $\zeta_{n\vec{p}\sigma}$ as

$$\zeta_{n\vec{p}\sigma} = \zeta_n(-)^{j-\sigma} \qquad \text{(standard phase convention)}\,. \tag{8.371}$$

where $(-)^j$ is pulled out of the dependence on the particle type so that the factor $(-)^{j-\sigma}$ becomes ± 1 even for the cases where j is half-integer.

To summarize, under time reversal, fields transform as

$$\boxed{\begin{array}{rcll} \mathcal{T}\phi(x)\mathcal{T}^\dagger &=& \zeta_n^*\phi(Tx) & (\text{spin}-0) \\ \mathcal{T}\psi(x)\mathcal{T}^\dagger &=& \zeta_n^*\mathrm{T}\psi(Tx) & (\text{spin}-\frac{1}{2}) \\ \mathcal{T}A^\mu(x)\mathcal{T}^\dagger &=& \zeta_n^* A_\mu(Tx) & (\text{spin}-1) \end{array}}\,, \tag{8.372}$$

where the phase factors ζ_n can differ for different particles, and T is a unitary and antisymmetric 4×4 matrix that satisfies $\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma_\mu^*$. Note that the Lorentz index μ changed its position for the spin-1 field.

Let's now find how $\mathcal{T}$ transforms the QED interaction

$$h_{QED}(t) = \int d^3x \mathcal{H}_{\text{int}}(x) = \int d^3x A^\mu(x) j_\mu(x)\,, \quad j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)\,. \tag{8.373}$$

Using (8.372), the transformation of the current under time reversal is (setting $n=f$)

$$\begin{aligned} \mathcal{T}j_\mu(x)\mathcal{T}^\dagger &= \underbrace{\mathcal{T}\psi^\dagger(x)\mathcal{T}^\dagger}_{(\zeta_f^*\mathrm{T}\psi(x'))^\dagger}\ \underbrace{\mathcal{T}\gamma^0\gamma_\mu\mathcal{T}^\dagger}_{\gamma^{0*}\gamma_\mu^*}\underbrace{\mathcal{T}\psi(x)\mathcal{T}^\dagger}_{\zeta_f^*\mathrm{T}\psi(x')} \\ &= \underbrace{\zeta_f\zeta_f^*}_{1}\,\psi^\dagger(x')\mathrm{T}^\dagger\gamma^{0*}\gamma_\mu^*\mathrm{T}\psi(x')\,, \end{aligned} \tag{8.374}$$

where we have carefully distinguished the antilinear Hilbert space operator $\mathcal{T}$ and the 4×4 matrix T which is linear by definition. For example, in $\mathcal{T}\gamma^0\gamma_\mu\mathcal{T}^\dagger = \gamma^{0*}\gamma_\mu^*$, all we have used is the antilinearity of $\mathcal{T}$ where $\mathcal{T}$ passed over the gamma matrices and annihilated with $\mathcal{T}^\dagger$. Sandwiching (8.336) between $\mathrm{T}^\dagger$ and T,

$$\gamma^\mu = \mathrm{T}^\dagger\gamma_\mu^*\mathrm{T}\,. \tag{8.375}$$

Thus,

$$\mathrm{T}^\dagger\gamma^{0*}\gamma_\mu^*\mathrm{T} = \mathrm{T}^\dagger\gamma^{0*}\mathrm{T}\mathrm{T}^\dagger\gamma_\mu^*\mathrm{T} = \gamma_0\gamma^\mu\,. \tag{8.376}$$

Then, the current transformation is

$$\mathcal{T}j_\mu(x)\mathcal{T}^\dagger = \psi^\dagger(x')\gamma_0\gamma^\mu\psi(x') = j^\mu(x')\,. \tag{8.377}$$

Namely, the space part change sign while the time component stays the same, which is what we expect for the current of electric charge. Combining this with the transformation of the photon field (with $n = v$),

$$\mathcal{T}\mathcal{H}_{\rm int}(x)\mathcal{T}^\dagger = \mathcal{T}A^\mu(x)\mathcal{T}^\dagger\mathcal{T}j_\mu(x)\mathcal{T}^\dagger = \zeta_v^* A_\mu(x')j^\mu(x')\,. \tag{8.378}$$

At this point, we choose the arbitrary phase ζ_v^* to be unity and obtain

$$\mathcal{T}\mathcal{H}_{\rm int}(x)\mathcal{T}^\dagger = A_\mu(x')j^\mu(x') = \mathcal{H}_{\rm int}(x') \quad (x' = Tx)\,. \tag{8.379}$$

The space-integrated interaction Hamiltonian then transforms as

$$\mathcal{T}h_{QED}(t)\mathcal{T}^\dagger = \int d^3x\mathcal{T}\mathcal{H}_{\rm int}(x)\mathcal{T}^\dagger = \int d^3x\mathcal{H}_{\rm int}(x') = h_{QED}(-t)\,. \tag{8.380}$$

Thus, all processes caused by the QED interaction Hamiltonian is invariant under time reversal as defined by (8.295).

Similarly, it is easy to show that the Yukawa coupling transforms under $\mathcal{T}$ as

$$\mathcal{T}\bar\psi(x)\psi(x)\,\phi(x)\mathcal{T}^\dagger = \zeta_s^*\bar\psi(x')\psi(x')\,\phi(x')\,, \tag{8.381}$$

and thus, by choosing $\zeta_s^* = 1$, we can make the interaction Hamiltonian satisfy $\mathcal{T}h_Y(t)\mathcal{T}^\dagger = h_Y(-t)$. Since we could choose the time-reversal phases such that this happens, all processes caused by the Yukawa coupling is invariant under time reversal.

Exercise 8.11 *Show that by choosing the time reversal phase of the spin-0 field in the Yukawa coupling interaction to be +1, the interaction becomes symmetric under time reversal.*

8.5.4 $\mathcal{T}^2$ and Kramer's degeneracy

For spin-0 and spin-1 fields, applying $\mathcal{T}$ twice recovers the original field:

$$\begin{aligned}\mathcal{T}\mathcal{T}\phi(x)\mathcal{T}^\dagger\mathcal{T}^\dagger &= \zeta_n\mathcal{T}\phi(Tx)\mathcal{T}^\dagger = \zeta_n\zeta_n^*\phi(T^2x) = \phi(x)\\ \mathcal{T}\mathcal{T}A^\mu(x)\mathcal{T}^\dagger\mathcal{T}^\dagger &= \zeta_n\mathcal{T}A_\mu(Tx)\mathcal{T}^\dagger = \zeta_n\zeta_n^*A^\mu(T^2x) = A^\mu(x)\,.\end{aligned} \tag{8.382}$$

For a spin-1/2 fields, however, applying $\mathcal{T}$ twice flips the sign:

$$\begin{aligned}\mathcal{T}\mathcal{T}\psi(x)\mathcal{T}^\dagger\mathcal{T}^\dagger &= \mathcal{T}\zeta_n^*\mathrm{T}\psi(Tx)\mathcal{T}^\dagger = \zeta_n\mathrm{T}^*\mathcal{T}\psi(Tx)\mathcal{T}^\dagger\\ &= \zeta_n\zeta_n^*\underbrace{\mathrm{T}^*\mathrm{T}}_{-1\text{ by (8.348)}}\psi(T^2x) = -\psi(x)\,.\end{aligned} \tag{8.383}$$

This originated from the σ-dependence of the T-reversal phase. Using the convention-independent form (8.356) for the time-reversal phase, action of two $\mathcal{T}$'s on a fermion creation operator is

$$\begin{aligned}\mathcal{T}\mathcal{T}a^\dagger_{f\vec{p}\sigma}\mathcal{T}^\dagger\mathcal{T}^\dagger &= \mathcal{T}\delta\, s_{-\sigma}\,\zeta_f a^\dagger_{f-\vec{p}-\sigma}\mathcal{T}^\dagger = \delta^* s_{-\sigma}\,\zeta_f^*\,\mathcal{T}a^\dagger_{f-\vec{p}-\sigma}\mathcal{T}^\dagger \\ &= \delta^* s_{-\sigma}\,\zeta_f^*\,\delta\, s_\sigma\,\zeta_f\; a^\dagger_{f\vec{p}\sigma} = |\delta|^2\,|\zeta_f|^2\, s_{-\sigma}s_\sigma\; a^\dagger_{f\vec{p}\sigma} \\ &= -a^\dagger_{f\vec{p}\sigma}\,, \end{aligned} \tag{8.384}$$

where we have used $s_{-\sigma}s_\sigma = -1$ which holds regardless of the value of $\sigma = \pm 1/2$. Thus, applying $\mathcal{T}$ twice on a fermion creation operator flips its sign.

For a particle of arbitrary spin, let's use the standard phase convention (8.371) for the time reversal. Noting that the factor $(-)^{j-\sigma}$ is always real,

$$\begin{aligned}\mathcal{T}\mathcal{T}a^\dagger_{n\,\vec{p}\sigma}\mathcal{T}^\dagger\mathcal{T}^\dagger &= \mathcal{T}\zeta_n(-)^{j-\sigma}a^\dagger_{n\,-\vec{p}-\sigma}\mathcal{T}^\dagger = \zeta_n^*(-)^{j-\sigma}\mathcal{T}a^\dagger_{n\,-\vec{p}-\sigma}\mathcal{T}^\dagger \\ &= \zeta_n^*(-)^{j-\sigma}\zeta_n(-)^{j+\sigma}a^\dagger_{n\,\vec{p}\sigma} = (-)^{2j}a^\dagger_{n\,\vec{p}\sigma}\,. \end{aligned} \tag{8.385}$$

Namely, applying $\mathcal{T}$ twice flips the sign of a creation operator for a half-integer spin in general. You may well be feeling uncomfortable, since the derivation above used the fact that the σ dependence of the time reversal phase was given by (8.371) which in turn was a result of the phase convention of the spin eigen states (8.366). As we will show below, the conclusion that $\mathcal{T}^2$ acts as $(-)^{2j}$ on a single-particle state does not depend on the phase convention of the spin eigen states.

Let's first define the spin eigen states of a particle at rest with arbitrary relative phases. Thus, instead of (8.366) we write

$$J_\pm|n\,\vec{0}\,\sigma\rangle = \delta^\pm_\sigma\sqrt{(j\mp\sigma)(j\pm\sigma+1)}\,|n\,\vec{0}\,\sigma{\pm}1\rangle\,, \tag{8.386}$$

where $\delta^\pm_\sigma$ is an arbitrary phase factor that depends on σ and whether the operator applied is the raising or lowering operator. For self-consistency, the phase factors $\delta^\pm_\sigma$ cannot be completely arbitrary. Applying the operator relation

$$J_+J_- = (J_x+iJ_y)(J_x-iJ_y) = J_x^2+J_y^2+\underbrace{i[J_y,J_x]}_{J_z} = J^2-J_z^2+J_z \tag{8.387}$$

to the state $|n\,\vec{0}\,\sigma\rangle$,

$$J_+J_-|n\,\vec{0}\,\sigma\rangle = (\underbrace{J^2}_{j(j+1)}-J_z^2+J_z)|n\,\vec{0}\,\sigma\rangle = \Big(j(j+1)-\sigma^2+\sigma\Big)\,|n\,\vec{0}\,\sigma\rangle\,. \tag{8.388}$$

On the other hand, using (8.386) we have

$$J_+J_-|n\,\vec{0}\,\sigma\rangle = \delta^-_\sigma\,\delta^+_{\sigma-1}\underbrace{\sqrt{(j+\sigma)(j-\sigma+1)}\sqrt{(j-\sigma+1)(j+\sigma)}}_{(j+\sigma)(j-\sigma+1)\,=\,j(j+1)-\sigma^2+\sigma}|n\,\vec{0}\,\sigma\rangle\,. \tag{8.389}$$

Comparing (8.389) and (8.389), we obtain

$$\delta^-_\sigma \delta^+_{\sigma-1} = 1 \quad \to \quad \delta^-_\sigma = \delta^{+*}_{\sigma-1}\,; \tag{8.390}$$

namely, the phase factor used when σ is lowered to $\sigma - 1$ should be the complex conjugate of the phase factor used when $\sigma - 1$ is raised to σ. Now, the state $|n\,\vec{0}\,\sigma\rangle$ can be obtained from $|n\,\vec{0}\,{-\sigma}\rangle$ by applying J_+ 2σ times: using (8.386),

$$(J_+)^{2\sigma}|n\,\vec{0}\,{-\sigma}\rangle = \prod_{i=-\sigma}^{\sigma-1}\delta^+_i \prod_{i=-\sigma}^{\sigma-1}\sqrt{(j-i)(j+i+1)}\;|n\,\vec{0}\,\sigma\rangle\,. \tag{8.391}$$

Applying $\mathcal{T}$ to this and using $\mathcal{T}J_+ = -J_-\mathcal{T}$ (8.368),

$$\underbrace{\underbrace{\mathcal{T}(J_+)^{2\sigma}}_{(-J_-)^{2\sigma}\mathcal{T}}\;|n\,\vec{0}\,{-\sigma}\rangle}_{(-)^{2\sigma}\zeta_{n\,\vec{0}\,-\sigma}\underbrace{(J_-)^{2\sigma}|n\,\vec{0}\,\sigma\rangle}_{\prod_{i=\sigma}^{-\sigma+1}\delta^-_i\prod_{i=\sigma}^{-\sigma+1}\sqrt{(j+i)(j-i+1)}\;|n\,\vec{0}\,{-\sigma}\rangle\;\leftarrow\;(8.386)}} = \prod_{i=-\sigma}^{\sigma-1}\delta^{+*}_i \prod_{i=-\sigma}^{\sigma-1}\sqrt{(j-i)(j+i+1)}\;\underbrace{\mathcal{T}|n\,\vec{0}\,\sigma\rangle}_{\zeta_{n\,\vec{0}\,\sigma}|n\,\vec{0}\,{-\sigma}\rangle}\,. \tag{8.392}$$

Relabeling the index as $i \to -i$, we have

$$\prod_{i=\sigma}^{-\sigma+1}\sqrt{(j+i)(j-i+1)} = \prod_{i=-\sigma}^{\sigma-1}\sqrt{(j-i)(j+i+1)}\,. \tag{8.393}$$

Using $\delta^-_\sigma = \delta^{+*}_{\sigma-1}$ and shifing the summation index by one, we have

$$\prod_{i=\sigma}^{-\sigma+1}\delta^-_i = \prod_{i=\sigma}^{-\sigma+1}\delta^{+*}_{i-1} = \prod_{i=\sigma-1}^{-\sigma}\delta^{+*}_i = \prod_{i=-\sigma}^{\sigma-1}\delta^{+*}_i\,. \tag{8.394}$$

Using (8.393) and (8.394) in (8.392), we obtain $(-)^{2\sigma}\zeta_{n\,\vec{0}\,-\sigma} = \zeta_{n\,\vec{0}\,\sigma}$, or since the time-reversal phase should not depend on $\vec{p}$, we can drop the momentum index:

$$(-)^{2\sigma}\zeta_{n\,-\sigma} = \zeta_{n\,\sigma} \tag{8.395}$$

Then, applying $\mathcal{T}$ twice on $a^\dagger_{n\,\vec{p}\,\sigma}$ yields

$$\mathcal{T}\;\underbrace{\mathcal{T}a^\dagger_{n\,\vec{p}\,\sigma}\mathcal{T}^\dagger}_{\zeta_{n\,\sigma}a^\dagger_{n\,-\vec{p}\,-\sigma}}\;\mathcal{T}^\dagger = \zeta_{n\,-\sigma}\zeta^*_{n\,\sigma}a^\dagger_{n\,\vec{p}\,\sigma} = (-)^{2\sigma}a^\dagger_{n\,\vec{p}\,\sigma}\,, \tag{8.396}$$

namely, $\mathcal{T}^2$ acts as $(-)^{2\sigma}$ on a single particle state, and this is independent of the phase convention of the spin eigen states. We then note that for any (j,σ), intgers or half-integers, we have $(-)^{2\sigma} = (-)^{2j}$. Thus, if a state contains an odd number of particles with half-integer spin, then $\mathcal{T}^2$ acts as -1. Does this have any physical consequences? It does, as we will now find.

Suppose that a state Ψ is an eigenstate of the total Hamiltonian H with a real eigenvalue E:

$$H|\Psi\rangle = E|\Psi\rangle\,, \tag{8.397}$$

which we assume to hold at any time t, and that the state is $\mathcal{T}^2$-odd:

$$\mathcal{T}^2|\Psi\rangle = -|\Psi\rangle\,; \tag{8.398}$$

namely, it contains an odd number of half-inter particles. Now assume that the interaction is invariant under time reversal. In the Schrödinger picture we have $\mathcal{T}H\mathcal{T}^\dagger = H$. Then, applying $\mathcal{T}$ to (8.397),

$$\mathcal{T}H\underbrace{\,|\Psi\rangle}_{\mathcal{T}^\dagger\mathcal{T}} = E\mathcal{T}|\Psi\rangle \quad \to \quad H\mathcal{T}|\Psi\rangle = E\mathcal{T}|\Psi\rangle\,. \tag{8.399}$$

Namely, the state $\mathcal{T}|\Psi\rangle$ also has the same energy E. On the other hand, using the definition (8.268),

$$(\mathcal{T}\Psi,\underbrace{\,\Psi}_{\mathcal{T}^\dagger\mathcal{T}}) = (\underbrace{\mathcal{T}^2\Psi}_{-\Psi},\mathcal{T}\Psi)^* = -(\mathcal{T}\Psi,\Psi) \quad \to \quad (\mathcal{T}\Psi,\Psi) = 0\,. \tag{8.400}$$

Thus, the two states $|\Psi\rangle$ and $\mathcal{T}|\Psi\rangle$ are orthogonal to each other while having the same energy. This is called Kramer's degeneracy.

If the system is spherically symmetric, the degeneracy is trivial since the energy for such a system is independent of the component m of the total angular momentum resulting in $(2j+1)$-fold degeneracy where j is the absolute value of the total angular momentum of the system. If the state contains an odd number of particles with half-integer spin, then the total angular momentum is half-integer, and thus there are at least two possible values for m.

When the system is in an external field that is T-invariant (an electric field, for example), then the Hamiltonian of the system (without the source of the external field) is still T-invariant, and the Kramer's degeneracy still holds; namely, for each state there is at least one other state with the same energy. What is remarkable is that such a degeneracy exists no matter how complicated the system and the external field may be as long as the interaction is T-invariant. When the external field changes sign under time reversal (a magnetic field, for example), then the effective Hamiltonian of the system (without including the source of the external field) is T-violating, and as a result $|\Psi\rangle$ and $\mathcal{T}|\Psi\rangle$ do not in general have the same energy. In such cases, the Kramer's degeneracy, which is a consequence of T-invariance, is removed. Note that we are concerned here with the time reversal symmetry of the Hamiltonian without including the source of the external field. If the source is included in the Hamiltonian, then the total Hamiltonian would be invariant under T even when the field is a magnetic field.

8.5.5 Applications of T-invariance

In the definition of the Lorentz-invariant matrix element $\mathcal{M}$ (5.107)

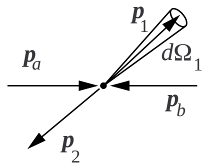
$$S_{fi} \equiv \frac{(2\pi)^4\delta^4\Big(\sum_k p_{i_k} - \sum_k p_{f_k}\Big)}{\sqrt{\prod_k(2p^0_{i_k}V)\prod_k(2p^0_{f_k}V)}}\,\mathcal{M}\,, \tag{8.401}$$

we note that the factor in front of $\mathcal{M}$ is invariant under $\vec{p} \leftrightarrow -\vec{p}$ for all momenta. Thus, in terms of $\mathcal{M}$, the reciprocity relation (8.237) can be written as

$$\mathcal{M}_{(n_f\,\vec{p}_f\,\sigma_f)\to(n_i\,\vec{p}_i\,\sigma_i)} = e^{i\phi}\mathcal{M}_{(n_i\,-\vec{p}_i\,-\sigma_i)\to(n_f\,-\vec{p}_f\,-\sigma_f)}\,, \tag{8.402}$$

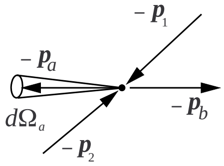
where ϕ is some phase angle. This may be applied to a 2-body scattering $a+b \to 1+2$ in the C.M. frame, where the scattering cross section is given by

$$\frac{d\sigma}{d\Omega_1}(ab \to 12) = \frac{|\mathcal{M}_{ab\to 12}|^2}{(8\pi M)^2}\frac{|\vec{p}_1|}{|\vec{p}_a|}\,. \tag{8.403}$$



On the other hand, the cross section for $1+2 \to a+b$ where all momenta and spins are sign-flipped is given by

$$\frac{d\sigma}{d\Omega_a}(12 \to ab) = \frac{|\mathcal{M}_{12\to ab}|^2}{(8\pi M)^2}\frac{|\vec{p}_a|}{|\vec{p}_1|}\,. \tag{8.404}$$



Then, the T-invariance (8.402) indicates

$$\frac{|\vec{p}_a|}{|\vec{p}_1|}\frac{d\sigma}{d\Omega_1}(ab \to 12) = \frac{|\vec{p}_1|}{|\vec{p}_a|}\frac{d\sigma}{d\Omega_a}(12 \to ab) \tag{8.405}$$

where it is understood that from the left-hand side to the right-hand side, all momenta and spins measured in the laboratory frame are sign-flipped. Note, however, that the corresponding scattering angles (θ, ϕ) for the particle 1 on the left-hand side and those for the particle a on the right-hand side are the same. Also, since momenta and spins are flipped, the helicities are kept the same. This can be checked experimentally for example in $e^+e^- \to \tau^+\tau^-$. All observed QED processes are consistent with the T-invariance. Similarly, all processes due to the strong interaction are found to be symmetric under time reversal.

When spins are not measured, we have to sum over all spins and divide by the total number of spin configurations in the initial state. Thus, the unpolarized cross sections are

$$\begin{aligned} \frac{d\sigma}{d\Omega_1}(ab \to 12) &= \frac{1}{(8\pi M)^2} \frac{\sum_{\text{spins}} |\mathcal{M}_{ab\to 12}|^2}{(2j_a+1)(2j_b+1)} \frac{|\vec{p}_1|}{|\vec{p}_a|} \\ \frac{d\sigma}{d\Omega_a}(12 \to ab) &= \frac{1}{(8\pi M)^2} \frac{\sum_{\text{spins}} |\mathcal{M}_{12\to ab}|^2}{(2j_1+1)(2j_2+1)} \frac{|\vec{p}_a|}{|\vec{p}_1|} \,. \end{aligned} \tag{8.406}$$

If $\mathcal{T}$ is invariant, we have $\sum_{\text{spins}} |\mathcal{M}_{ab\to 12}|^2 = \sum_{\text{spins}} |\mathcal{M}_{12\to ab}|^2$, and thus

$$\frac{d\sigma/d\Omega_1(ab \to 12)}{d\sigma/d\Omega_a(12 \to ab)} = \frac{(2j_1+1)(2j_2+1)}{(2j_a+1)(2j_b+1)} \frac{\vec{p}_1^{\,2}}{\vec{p}_a^{\,2}} \,. \tag{8.407}$$

This relation is called the 'detailed balance', and was applied, among others, to the reaction

$$\pi^+ + d \leftrightarrow p + p \qquad (d = \text{deuteron},\ j = 1;\ p = \text{proton},\ j = \tfrac{1}{2}) \tag{8.408}$$

to determine the spin of pion, which was found to be zero. Note that when the cross sections are integrated over production angles and there are identical particles in the final state, then the expected rate should be divided by 2 after integrating over 4π steradian. Namely,

$$\frac{\sigma(ab \to 12)}{\sigma(12 \to ab)} = (\text{symmetry factor}) \times \frac{(2j_1+1)(2j_2+1)}{(2j_a+1)(2j_b+1)} \frac{\vec{p}_1^{\,2}}{\vec{p}_a^{\,2}} \,, \tag{8.409}$$

where the symmetry factor when particles 1 and 2 are identical is 1/2.

8.6 Bilinear covariants under $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$

The transformation properties of fields given in (8.96), (8.189), and (8.372) are all needed to find how a given Lagrangian term transforms under $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$. If one can choose the phases that appear in the transformation rules to make the interaction Hamiltonian invariant under one of, or a combination of, these transformations, then the corresponding physics is invariant under such transformation. Fermion fields, however, always appear as bilinear covariants in interaction Lagrangian terms, and it

	current	$\mathcal{P}(x' = Px)$	$\mathcal{C}(x' = x)$	$\mathcal{T}(x' = Tx)$
(S)	$\bar{\psi}_1\psi_2$	$\eta_1\eta_2^*\bar{\psi}_1\psi_2$	$\xi_1\xi_2^*\bar{\psi}_2\psi_1$	$\zeta_1\zeta_2^*\bar{\psi}_1\psi_2$
(V)	$\bar{\psi}_1\gamma_\mu\psi_2$	$\eta_1\eta_2^*\bar{\psi}_1\gamma^\mu\psi_2$	$-\xi_1\xi_2^*\bar{\psi}_2\gamma_\mu\psi_1$	$\zeta_1\zeta_2^*\bar{\psi}_1\gamma^\mu\psi_2$
(T)	$\bar{\psi}_1\sigma_{\mu\nu}\psi_2$	$\eta_1\eta_2^*\bar{\psi}_1\sigma^{\mu\nu}\psi_2$	$-\xi_1\xi_2^*\bar{\psi}_2\sigma_{\mu\nu}\psi_1$	$-\zeta_1\zeta_2^*\bar{\psi}_1\sigma^{\mu\nu}\psi_2$
(A)	$\bar{\psi}_1\gamma_\mu\gamma_5\psi_2$	$-\eta_1\eta_2^*\bar{\psi}_1\gamma^\mu\gamma_5\psi_2$	$\xi_1\xi_2^*\bar{\psi}_2\gamma_\mu\gamma_5\psi_1$	$\zeta_1\zeta_2^*\bar{\psi}_1\gamma^\mu\gamma_5\psi_2$
(P)	$\bar{\psi}_1\gamma_5\psi_2$	$-\eta_1\eta_2^*\bar{\psi}_1\gamma_5\psi_2$	$\xi_1\xi_2^*\bar{\psi}_2\gamma_5\psi_1$	$\zeta_1\zeta_2^*\bar{\psi}_1\gamma_5\psi_2$

Table 8.1: Transformation of bilinear covariants under $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$. The transformed currents are functions of x'. Note that Lorentz indices for $\mathcal{P}$ and $\mathcal{T}$ change from subscripts to superscripts, and that ψ_1 and ψ_2 were swapped for $\mathcal{C}$.

is convenient to have a list of their transformations under $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$. We write a bilinear covariant as

$$j(x) = \bar{\psi}_1(x)M\psi_2(x)\,, \tag{8.410}$$

where ψ_1 and ψ_2 are two fermion fields and the 4×4 matrix M is one of Γ_i's in Table 3.2.

Using the parity transformation of a fermion field (8.96) and its spinor adjoint

$$\mathcal{P}\psi(x)\mathcal{P}^\dagger = \eta^*\gamma^0\psi(x') \quad \to \quad \mathcal{P}\bar{\psi}(x)\mathcal{P}^\dagger = \eta\bar{\psi}(x')\gamma^0 \quad (x' \equiv Px)\,, \tag{8.411}$$

the bilinear covariant transforms under $\mathcal{P}$ as

$$\begin{aligned}\mathcal{P}\,\bar{\psi}_1(x)M\psi_2(x)\,\mathcal{P}^\dagger &= \underbrace{\mathcal{P}\bar{\psi}_1(x)\mathcal{P}^\dagger}_{\eta_1\bar{\psi}_1(x')\gamma^0}\ M\ \underbrace{\mathcal{P}\psi_2(x)\mathcal{P}^\dagger}_{\eta_2^*\gamma^0\psi_2(x')}\\ &= \eta_1\eta_2^*\,\bar{\psi}_1(x')(\gamma^0M\gamma^0)\psi_2(x') \quad (x' \equiv Px)\,.\end{aligned} \tag{8.412}$$

Using the relations

$$\begin{gathered}\gamma^0 I\gamma^0 = I\,, \quad \gamma^0\gamma_\mu\gamma^0 = \gamma^\mu\,, \quad \gamma^0 i\gamma_\mu\gamma_\nu\gamma^0 = i\gamma^\mu\gamma^\nu\\ \gamma^0\gamma_\mu\gamma_5\gamma^0 = -\gamma^\mu\gamma_5\,, \quad \gamma^0\gamma_5\gamma^0 = -\gamma_5\,,\end{gathered} \tag{8.413}$$

one obtains the parity transformations listed in Table 8.1.

For charge conjugation, the fermion transformation (8.189) and its Hermitian conjugate are

$$\mathcal{C}\psi\mathcal{C}^\dagger = \xi^*\Gamma\psi^\dagger \quad \xrightarrow{\dagger} \quad \mathcal{C}\psi^\dagger\mathcal{C}^\dagger = \xi\psi^T\Gamma^\dagger\,. \tag{8.414}$$

Recalling that normal ordering has been implicit, the current $\bar{\psi}M\psi$ transforms under $\mathcal{C}$ as

$$\begin{aligned} :\mathcal{C}\bar{\psi}_1 M\psi_2\mathcal{C}^\dagger: &= :\underbrace{\mathcal{C}\psi_1^\dagger\mathcal{C}^\dagger}_{\xi_1\psi_1^T\Gamma^\dagger}\ \gamma^0 M\ \underbrace{\mathcal{C}\psi_2\mathcal{C}^\dagger}_{\xi_2^*\Gamma\psi_2^*}: \\ &= \xi_1\xi_2^* :\psi_1^T\,(\Gamma^\dagger\gamma^0 M\,\Gamma)\,\psi_2^*: \end{aligned} \tag{8.415}$$

For $M = I$ (S), γ^μ (V), $i\gamma^\mu\gamma^\nu(\mu<\nu)$ (T), $\gamma^\mu\gamma_5$ (A), and γ_5 (P), the relation (8.174) straightforwardly leads to

$$\Gamma^\dagger\gamma^0 M\,\Gamma = s_C\,(\gamma^0 M)^T \tag{8.416}$$

with the sign s_C given by

$$s_C = -1\ (S),\ +1\ (V),\ +1\ (T),\ -1\ (A),\ -1\ (P)\,. \tag{8.417}$$

Then, we have

$$\begin{aligned} :\mathcal{C}\bar{\psi}_1 M\psi_2\mathcal{C}^\dagger: &= s_C\,\xi_1\xi_2^* :\psi_1^T\,(\gamma^0 M)^T\,\psi_2^*: \\ &= s_C\,\xi_1\xi_2^* :\psi_{1i}\,(\gamma^0 M)_{ji}\,\psi_{2j}^\dagger: \\ (\psi_{1i}\leftrightarrow\psi_{2j}^\dagger)\qquad &= -s_C\,\xi_1\xi_2^* :\psi_{2j}^\dagger\,(\gamma^0 M)_{ji}\,\psi_{1i}: \\ &= -s_C\,\xi_1\xi_2^* :\bar{\psi}_2 M\psi_1:\,, \end{aligned} \tag{8.418}$$

which is the column for $\mathcal{C}$ in Table 8.1.

The time reversal transformation of a fermion field (8.372) and its Hermitian conjugate are

$$\mathcal{T}\psi(x)\mathcal{T}^\dagger = \zeta^*\mathrm{T}\psi(x') \quad \xrightarrow{\dagger} \quad \mathcal{T}\psi^\dagger(x)\mathcal{T}^\dagger = \zeta\psi^\dagger(x')\mathrm{T}^\dagger \quad (x'\equiv Tx)\,. \tag{8.419}$$

Paying attention to the antilinearity of the operator $\mathcal{T}$, the current $\bar{\psi}_1 M\psi_2$ transforms under $\mathcal{T}$ as

$$\begin{aligned} \mathcal{T}\bar{\psi}_1(x)M\psi_2(x)\mathcal{T}^\dagger &= \underbrace{\mathcal{T}\psi_1^\dagger(x)(\mathcal{T}^\dagger}_{\zeta_1\psi^\dagger(x')\mathrm{T}^\dagger}\underbrace{\mathcal{T})\gamma^0 M(\mathcal{T}^\dagger}_{\gamma^{0*}M^*}\underbrace{\mathcal{T})\psi_2(x)\mathcal{T}^\dagger}_{\zeta_2^*\mathrm{T}\psi_2(x')} \\ &= \zeta_1\zeta_2^*\,\psi_1^\dagger(x')(\mathrm{T}^\dagger\gamma^{0*}M^*\mathrm{T})\psi_2(x') \quad (x'\equiv Tx)\,. \end{aligned} \tag{8.420}$$

Using $\gamma^\mu = \mathrm{T}^\dagger\gamma_\mu^*\mathrm{T}$ (8.375), it is easy to show that, for each type of M, we have

$$\mathrm{T}^\dagger\gamma^{0*}M^{\mu_1\cdots\mu_n *}\,\mathrm{T} = s_T\,\gamma^0 M_{\mu_1\cdots\mu_n}\,, \tag{8.421}$$

where we have explicitly written the Lorentz indices and the sign s_T is given by

$$s_T = +1\ (S),\ +1\ (V),\ -1\ (T),\ +1\ (A),\ +1\ (P)\,. \tag{8.422}$$

Then, the time-reversal transformation of the bilinear covariants becomes

$$\mathcal{T}\bar{\psi}_1(x)M^{\mu_1\cdots\mu_n}\psi_2(x)\mathcal{T}^\dagger = s_T\,\zeta_1\zeta_2^*\,\bar{\psi}_1(x')M_{\mu_1\cdots\mu_n}\psi_2(x') \tag{8.423}$$

which is given in Table 8.1.

8.7 CPT theorem

One of the most fundamental theorems in quantum field theory is the CPT theorem which states that, in general, the interaction Hamiltonian density is invariant under the combination $\mathcal{CPT}$ in the sense that CPT phases of fields can be adjusted such that

$$\Omega\mathcal{H}_{\rm int}(x)\Omega^\dagger = \mathcal{H}_{\rm int}(-x)\,, \qquad \Omega \equiv \mathcal{CPT} \text{ (antiunitary)} \tag{8.424}$$

holds, where the space-time argument sign-flipped for all of the four components: $PTx = -x$. The importance of the CPT theorem mainly comes from its generality: it is based on the assumptions of (a) microscopic causality, (b) Lorentz invariance, and (c) the continuity of the field operators. In fact, these are the basic principles upon which we have built the theory of quantized fields. We are well equipped to prove the CPT theorem so let's proceed.

8.7.1 Proof of the CPT theorem

Here, we will prove the CPT theorem for a general interaction involving spin-0, 1/2 and 1 fields. Let us first find out how each fields transform under CPT. Using the transformations under $\mathcal{P}$ (8.96), under $\mathcal{C}$ (8.189) and under $\mathcal{T}$ (8.372), we obtain

$$\begin{aligned}\Omega\phi(x)\Omega^\dagger &= \mathcal{CPT}\phi(x)\mathcal{T}^\dagger\mathcal{P}^\dagger\mathcal{C}^\dagger \\ &= \zeta_n^*\eta_n^*\xi_n^*\,\phi^\dagger(-x)\,, \end{aligned} \tag{8.425}$$

$$\begin{aligned}\Omega\psi(x)\Omega^\dagger &= \mathcal{CP}\,\underbrace{\mathcal{T}\psi(x)\mathcal{T}^\dagger}_{\zeta_n^*\mathrm{T}\psi(Tx)}\,\mathcal{P}^\dagger\mathcal{C}^\dagger = \zeta_n^*\mathrm{T}\,\mathcal{C}\,\underbrace{\mathcal{P}\psi(Tx)\mathcal{P}^\dagger}_{\eta_n^*\gamma^0\psi(-x)}\,\mathcal{C}^\dagger \\ &= \zeta_n^*\mathrm{T}\eta_n^*\gamma^0\,\underbrace{\mathcal{C}\psi(-x)\mathcal{C}^\dagger}_{\xi_n^*\Gamma\psi^*(-x)} = \zeta_n^*\eta_n^*\xi_n^*\,(\mathrm{T}\gamma^0\Gamma)\psi^*(-x)\,, \end{aligned} \tag{8.426}$$

$$\begin{aligned}\Omega A^\mu(x)\Omega^\dagger &= \mathcal{CPT}A^\mu(x)\mathcal{T}^\dagger\mathcal{P}^\dagger\mathcal{C}^\dagger \\ &= -\zeta_n^*\eta_n^*\xi_n^*\,A^{\mu\dagger}(-x)\,. \end{aligned} \tag{8.427}$$

On account of $\Gamma\gamma_\mu^*\Gamma^\dagger = -\gamma_\mu$ and $\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma_\mu^*$, we have

$$\begin{aligned}(\mathrm{T}\gamma^0\Gamma)\gamma_\mu^*(\mathrm{T}\gamma^0\Gamma)^\dagger &= \mathrm{T}\gamma^0\,\underbrace{\Gamma\gamma_\mu^*\Gamma^\dagger}_{-\gamma_\mu}\,\gamma^0\mathrm{T}^\dagger \\ &= -\mathrm{T}\,\underbrace{\gamma^0\gamma_\mu\gamma^0}_{\gamma^\mu}\,\mathrm{T}^\dagger = -\gamma_\mu^*\,, \end{aligned} \tag{8.428}$$

namely,

$$\{(\mathrm{T}\gamma^0\Gamma)^*, \gamma_\mu\} = 0 \qquad (\mu = 0,1,2,3)\,. \tag{8.429}$$

It is straightforward to show that a matrix that anticommutes with all γ_μ should be γ_5 up to a constant which in this case can be taken as a phase factor (because both $\mathrm{T}\gamma^0\Gamma$ and γ_5 are unitary as can be easily shown):

$$(\mathrm{T}\gamma^0\Gamma)^* \propto \gamma_5 \quad \to \quad \mathrm{T}\gamma^0\Gamma = \delta\gamma_5^*\,, \tag{8.430}$$

where δ is some phase factor which can be absorbed into the arbitrary CPT phases $\zeta_n\eta_n\xi_n$ of fermion fields. Thus, we can write the transformations as

$$\boxed{\begin{array}{rcll} \Omega\phi(x)\Omega^\dagger &=& \omega_n^*\,\phi^\dagger(-x) & (\text{spin}-0) \\ \Omega\psi(x)\Omega^\dagger &=& \omega_n^*\,\gamma_5^*\psi^*(-x) & (\text{spin}-\frac{1}{2}) \\ \Omega A^\mu(x)\Omega^\dagger &=& -\omega_n^*\,A^{\mu\dagger}(-x) & (\text{spin}-1) \end{array}} \quad (\Omega\equiv\mathcal{CPT})\,, \tag{8.431}$$

where $\omega_n \equiv \zeta_n\eta_n\xi_n$.

Exercise 8.12 *Matrix that anticommutes with γ_μ.*
Suppose a 4×4 matrix M anticommutes with all γ_μ

$$\{M,\gamma_\mu\} = 0 \qquad (\mu = 0,1,2,3)\,. \tag{8.432}$$

Show that M is proportional to γ_5. (hint: Expand M in terms of the 16 independent matrices that are formed by gamma matrices (S,V,T,A,P), and show that all coefficients except the one for P are zero.)

Now, we will show that by taking the CPT phases to be

$$\omega_n = +1\,. \tag{8.433}$$

for all fields, one can accomplish $\Omega\mathcal{H}_{\text{int}}(x)\Omega^\dagger = \mathcal{H}_{\text{int}}(-x)$ for a general interaction. Consider a tensor quantity $T^{\mu_1\cdots\mu_n}(x)$ constructed out of spin-0, spin-1/2, and spin-1 fields through sums and products and derivatives ∂_μ. There can be multiple types of particles for each spin, and normal ordering is implicit. Some of the Lorentz indices may be due to the derivative ∂_μ. As we have seen in (8.300), $\mathcal{T}$ and ∂_0 commutes. Also, $\mathcal{T}$ commutes with the space derivative, thus $\mathcal{T}$ commutes with ∂_μ. On the other hand, $\mathcal{C}$ clearly commutes with ∂_μ since it does not change the space-time argument, and we have seen in (8.92) that $\mathcal{P}$ commutes with ∂_μ. Thus, $\mathcal{CP}$ commutes with ∂_μ, so Ω commutes with ∂_μ. Then, with $\omega_n = 1$, $\partial_\mu\phi(x)$ transforms under Ω as

$$\Omega(\partial_\mu\phi(x))\Omega^\dagger = \partial_\mu(\Omega\phi(x)\Omega^\dagger) = \partial_\mu(\phi^\dagger(-x)) = -(\partial_\mu\phi^\dagger)(-x)\,. \tag{8.434}$$

Similarly,

$$\Omega(\partial_\nu A^\mu(x))\Omega^\dagger = \partial_\nu(\Omega A^\mu(x)\Omega^\dagger) = -\partial_\nu(A^{\mu\dagger}(-x)) = (-)^2(\partial_\nu A^{\mu\dagger})(-x)\,. \tag{8.435}$$

Namely, since the space-time argument changes sign under Ω, the derivative ∂_μ effectively picks up a minus sign. Suppose $T^{\mu_1\cdots\mu_n}(x)$ consists of spin-0 and spin-1 fields and their derivatives only (namely, no fermion fields). All fields will turn into their hermitian conjugate under Ω. The Lorentz indices should in general come from $A^\mu(x)$, ∂^μ, or $\epsilon_{\alpha\beta\gamma\delta}$. If the Lorentz index is due to $A^\mu(x)$ or ∂^μ, each index picks up one minus sign under Ω. Since all numerical constants will become their complex conjugate under Ω due to antilinearity, we have even when $T^{\mu_1\cdots\mu_n}$ contains complex coupling constants

$$\Omega T^{\mu_1\cdots\mu_n}(x)\Omega^\dagger = (-)^n\Big(T^{\mu_1\cdots\mu_n}(-x)\Big)^\dagger, \tag{8.436}$$

where we noted that $A^\dagger$'s and $\phi^\dagger$'s all commute, thus the ordering of the fields does not matter inside the implicit normal ordering. The asymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ is just a real number; thus,

$$\Omega\epsilon_{\alpha\beta\gamma\delta}\Omega^\dagger = \epsilon_{\alpha\beta\gamma\delta}\,, \tag{8.437}$$

which also satisfies the transformation rule (8.436) since we have $n = 4$ (even).

Fermion fields always appear as bilinear covariants in the interaction and there are only five types of them as shown in Table 3.2. If we write a bilinear covariant as

$$j^{\mu_1\cdots\mu_n}(x) = \bar a(x)M^{\mu_1\cdots\mu_n}b(x)\,, \tag{8.438}$$

where $a(x)$ and $b(x)$ are fermion fields, then we have

type	S	V	T	A	P
$M^{\mu_1\cdots\mu_n}$	1	γ^μ	$\sigma^{\mu\nu}$	$\gamma_5\gamma^\mu$	γ_5
$\#(\gamma^\mu)$	0	1	2	3	4
$n \equiv \#(\text{index})$	0	1	2	1	0

. (8.439)

For a given bilinear covariant in the table, we noted the number of gamma matrices and the number of Lorentz indices which are seen to be both even or both odd; namely,

$$(-)^{\#\gamma^\mu} = (-)^n\,. \tag{8.440}$$

Using(8.431) with $\omega_n = 1$ and its Hermitian conjugate

$$\Omega\psi(x)\Omega^\dagger = \gamma_5^*\psi^*(-x)\,, \qquad \Omega\psi^\dagger(x)\Omega^\dagger = \psi^T(-x)\gamma_5^T\,. \tag{8.441}$$

Then, the bilinear form $j^{\mu_1\cdots\mu_n} = \bar a M^{\mu_1\cdots\mu_n}b$ transforms as

$$\begin{aligned}\Omega\, j^{\mu_1\cdots\mu_n}(x)\,\Omega^\dagger &= (\Omega a^\dagger(x)\Omega^\dagger)\Omega(\gamma^0 M^{\mu_1\cdots\mu_n})\Omega^\dagger(\Omega b(x)\Omega^\dagger)\\ &= a(-x)\underbrace{\gamma_5^T(\gamma^0 M^{\mu_1\cdots\mu_n})^*\gamma_5^*}_{(\gamma_5^\dagger\gamma^0 M^{\mu_1\cdots\mu_n}\gamma_5)^*}b^*(-x)\,.\end{aligned} \tag{8.442}$$

Using $\gamma_5^\dagger = \gamma_5$ and the fact that γ_5 anticommutes with γ^μ, we have

$$\gamma_5^\dagger \gamma^0 M^{\mu_1\cdots\mu_n}\gamma_5 = (-)^{n+1}\gamma^0 M^{\mu_1\cdots\mu_n}\,, \tag{8.443}$$

where we have used the fact that the even-oddness of the number of Lorentz indices is the same as that of the number of γ^μ's in $M^{\mu_1\cdots\mu_n}$ (8.440). Then, recalling the normal ordering, the transformation of bilinear covariant under Ω becomes

$$\begin{aligned} :\Omega\, j^{\mu_1\cdots\mu_n}(x)\,\Omega^\dagger: &= (-)^{n+1} : a(-x)(\gamma^0 M^{\mu_1\cdots\mu_n})^* b^*(-x) : \\ &= (-)^{n+1} : a_i(-x)(\gamma^0 M^{\mu_1\cdots\mu_n})^\dagger_{ji} b_j^\dagger(-x) : \\ &= (-)^{n} : b_j^\dagger(-x)(\gamma^0 M^{\mu_1\cdots\mu_n})^\dagger_{ji} a_i(-x) : \\ &= (-)^{n} : b^\dagger(-x)(\gamma^0 M^{\mu_1\cdots\mu_n})^\dagger a(-x) : \\ &= (-)^{n} : \Big(a^\dagger(-x)\gamma^0 M^{\mu_1\cdots\mu_n} b(-x)\Big)^\dagger : ; \end{aligned} \tag{8.444}$$

or,

$$\Omega j^{\mu_1\cdots\mu_n}(x)\Omega^\dagger = (-)^n \Big(j^{\mu_1\cdots\mu_n}(-x)\Big)^\dagger\,. \tag{8.445}$$

Namely, a bilinear covariant transforms exactly the same way as the tensor $T^{\mu_1\cdots\mu_n}(x)$ that is made of spin-0 and spin-1 fields - it changes to its hermitian conjugate and picks up a sign $(-)^n$. When there are a product of bilinear covariants, taking the hermitian conjugate of the whole changes the order of the bilinear covariants. Because of the normal ordering, one can reorder them at will and the sign is plus since there are two fermion fields in a bilinear covariant. Thus, the transformation (8.436) is valid when fermion fields are involved. Thus, (8.436) is valid when the interaction consists of any number of spin-0, spin-1/2, spin-1 fields, any number of ∂_μ's, and any number of $\epsilon_{\alpha\beta\gamma\delta}$'s.

Now the Hamiltonian density $\mathcal{H}(x)$ is the $(0,0)$ component of the energy-momentum tensor $J^{\mu\nu}(x)$ $(n=2)$, and it is hermitian; thus applying the transformation (8.436) to $\mathcal{H}(x) = J^{00}(x)$, we obtain

$$\Omega\mathcal{H}(x)\Omega^\dagger = \mathcal{H}(-x)\,. \tag{8.446}$$

Since the free field part of $\mathcal{H}(x)$ is also the $(0,0)$-component of a second-rank tensor and hermitian, thus

$$\Omega\mathcal{H}_{\rm int}(x)\Omega^\dagger = \mathcal{H}_{\rm int}(-x)\,. \tag{8.447}$$

This concludes the proof of the CPT theorem. We have closely followed the original proof by Lüders, and the theorem is sometimes called the Pauli-Lüders theorem. Later, a more rigorous and general proof was given in the framework of the axiomatic field theory by Jost, Dyson, and others.

8.7.2 Some consequences of the CPT theorem

Let us examine the physical consequences of the CPT theorem. Integrating (8.447) over space, the interaction Hamiltonian $h(t)$ transforms as

$$\Omega h(t)\Omega^\dagger = h(-t)\,, \tag{8.448}$$

which has the same form as the time reversal case (8.295). Then, following the same procedure as in the case of time reversal, we obtain

$$\boxed{\Omega S\Omega^\dagger = S^\dagger \quad (CPT \text{ invariance})}\,. \tag{8.449}$$

The S matrix element between an initial state Ψ_i and Ψ_f is

$$\begin{aligned} S_{f,i} &= (\Psi_f, \overbrace{}^{\Omega^\dagger\Omega} S \overbrace{}^{\Omega^\dagger\Omega} \Psi_i) = (\Psi_f, \Omega^\dagger \underbrace{\Omega S\Omega^\dagger}_{S^\dagger} \Omega\Psi_i) \\ &= (\Omega\Psi_f, S^\dagger\Omega\Psi_i)^* = (\Omega\Psi_i, S\Omega\Psi_f)\,; \end{aligned} \tag{8.450}$$

namely,

$$\boxed{S_{f,i} = S_{\Omega i,\Omega f} \quad (CPT \text{ invariance})}\,. \tag{8.451}$$

where

$$|\Omega i\rangle \equiv \Omega\Psi_i\,, \quad |\Omega f\rangle \equiv \Omega\Psi_f\,. \tag{8.452}$$

Under $\Omega = \mathcal{CPT}$, a multi-particle state changes as

$$\{n_i\,\vec{p}_i\,\sigma_i\} \overset{\mathcal{T}}{\rightarrow} \{n_i\; -\vec{p}_i\; -\sigma_i\} \overset{\mathcal{P}}{\rightarrow} \{n_i\,\vec{p}_i\; -\sigma_i\} \overset{\mathcal{C}}{\rightarrow} \{\bar{n}_i\,\vec{p}_i\; -\sigma_i\}\,. \tag{8.453}$$

Thus, the CPT invariance relation (8.451) means that if all spins are flipped, particle and antiparticles are exchanged, and initial and final state are interchanged, then the transition amplitude stays the same (up to a phase). Note that the momenta are not sign-flipped, and the spin quantization axis is fixed in the laboratory frame; i.e., σ is not a helicity.

One of the important consequences of the CPT symmetry is that a particle and its antiparticle have the same mass. The mass of a particle is given by the expectation value of the total Hamiltonian for the state with one such particle at rest. The total Hamiltonian can be written as the sum of the free part and the interaction part:

$$H = H_0 + h\,, \tag{8.454}$$

where H_0 is the bare unrenormalized Hamiltonian in which the masses are the bare masses and not the physical masses. One always input the same bare mass for each particle and its antiparticle in H_0. What is non-trivial is that the physical mass of a particle, after it dresses up with clouds of other particles that interact with it, is still the same as the physical mass of its antiparticle. Integrating (8.446) over space, we have

$$\Omega H(t)\Omega^\dagger = H(-t)\,. \tag{8.455}$$

Note that the total Hamiltonian in general depend on time in the interaction picture. A state of a physical single-particle at rest, denoted as Ψ, would have a cloud of other particles around it which modifies its mass, and it also may decay. The mass, however, should not depend on time, so we choose to evaluate at $t = 0$. Using $\Omega H(t)\Omega^\dagger = H(-t)$,

$$\begin{aligned} m_\Psi &\equiv (\underbrace{\Psi,}_{\Omega^\dagger\Omega}\ H(0)\ \underbrace{\Psi}_{\Omega^\dagger\Omega}) = (\Omega\Psi, H(0)\Omega\Psi)^* \\ &= (\Omega\Psi, H(0)\Omega\Psi) \equiv m_{\Omega\Psi}\,. \end{aligned} \tag{8.456}$$

The state $\Omega\Psi$ is obtained from Ψ by exchanging particle and antiparticle and flipping spin, and is identified as the antiparticle of Ψ. The mass should not depend on the spin component, and thus we conclude that the masses of particle and antiparticle are identical. This is true to all orders in perturbation theory or even if the effect is non-perturbative.

Another powerful prediction of the CPT invariance is the equal life times of particle and antiparticle. If we set $i = f$ in (8.451), we have

$$S_{i,i} = S_{\Omega i,\,\Omega i}. \tag{8.457}$$

For a single-particle state at rest, the transition probability $|S_{ii}|^2$ is the probability to find the same state as the final state after some (large) time duration T. Thus, the state $|i\rangle$ has the same decay rate as the state $|\Omega i\rangle$. Since the direction of the spin, or the spin component σ, should not affect the total decay rate due to the rotational symmetry, the CPT invariance indicates that the life time of a particle be the same as that of its antiparticle. This is again true to all orders.

In cases where the interaction can be divided into a large part and a small part, one can go further on what one can say about the decay rates of particle and antiparticle. The exact meaning of large and small will be clarified later. Suppose the interaction Hamiltonian can be written as the sum of the strong interaction and the weak interaction:

$$\mathcal{H}_{\rm int} = \mathcal{H}^s + \mathcal{H}^w\,. \tag{8.458}$$

Let S be the S-operator of the whole interaction and S^s be the S-operator that would result if there were only $\mathcal{H}^s$; then, both S and S^s are unitary:

$$S^\dagger S = 1\,, \quad S^{s\dagger} S^s = 1\,. \tag{8.459}$$

We then define σ as the difference between the two S operators:

$$S = S^s + \sigma\,. \tag{8.460}$$

The operator σ is thus whatever is added by the existence of $\mathcal{H}^w$ and includes effects of $\mathcal{H}^s$ and $\mathcal{H}^w$ at higher orders. The CPT invariance (8.449) should apply to S as well as to S^s:

$$\Omega S \Omega^\dagger = S^\dagger\,, \quad \Omega S^s \Omega^\dagger = S^{s\dagger}\,, \tag{8.461}$$

thus,

$$\Omega\,\sigma\,\Omega^\dagger = \sigma^\dagger\,. \tag{8.462}$$

Then, repeating the same derivation as in (8.450), we obtain,

$$\sigma_{f,i} = \sigma_{\Omega f,\Omega i}\,. \tag{8.463}$$

Now, the unitarities of S and S^s gives

$$\begin{aligned} \underbrace{S^\dagger S}_{1} &= (S^s+\sigma)^\dagger(S^s+\sigma) \\ &= \underbrace{S^{s\dagger}S^s}_{1} + \sigma^\dagger S^s + S^{s\dagger}\sigma + \sigma^\dagger\sigma\,. \end{aligned} \tag{8.464}$$

Taking the matrix elements between $|f\rangle$ and $|i\rangle$ and inserting all possible intermediate states $|m\rangle\langle m|$,

$$\sigma^\dagger_{f,m}S^s_{m,i} + S^{s\dagger}_{f,m}\sigma_{m,i} + \sigma^\dagger_{f,m}\sigma_{m,i} = 0\,, \tag{8.465}$$

where sum over m is implicit. At this point, we assume that the last term is much smaller than the first two, which defines the meaning of $\mathcal{H}^s$ being much larger than $\mathcal{H}^w$. Then we have

$$S^{s*}_{m,f}\sigma_{m,i} = -\sigma^*_{\Omega f,\Omega m}S^s_{m,i}\,, \tag{8.466}$$

where we have used

$$S^{s\dagger}_{f,m} = S^{s*}_{m,f} \quad \text{and} \quad \sigma^\dagger_{f,m} = \sigma^*_{m,f} = \sigma^*_{\Omega f,\Omega m} \;\leftarrow (8.463)\,. \tag{8.467}$$

In the above, m runs over all states; now suppose that S^s has nonzero matrix element only within certain subspaces; namely,

$$S^s_{m,i} = 0 \text{ if } m \notin \{i\}\,, \tag{8.468}$$

where $\{i\}$ are the set of all basis states that can be reached from the state i by the strong interaction. The basis of the subspace is complete in the sense that S^s is unitary within the subspace:

$$S^{s*}_{i,i'}S^s_{i,i"} = \delta_{i',i"} \quad (i,i',i" \in \{i\})\,, \tag{8.469}$$

where sum over i is implicit. Then, (8.466) can be written as

$$S^{s*}_{f',f}\sigma_{f',i} = -\sigma^*_{\Omega f,\Omega i'}S^s_{i',i} \qquad (i' \in \{i\}\,,\; f' \in \{f\})\,. \tag{8.470}$$

Squaring both sides, summing over $i \in \{i\}$ and $f \in \{f\}$, and using the unitarity of S^s in each subspace (8.469),

$$\underbrace{S^{s*}_{f',f} S^{s}_{f",f}}_{\delta_{f',f"}} \sigma_{f',i} \sigma^*_{f",i} = \sigma^*_{\Omega f,\Omega i'} \sigma_{\Omega f,\Omega i"} \underbrace{S^{s}_{i',i} S^{s*}_{i",i}}_{\delta_{i',i"}} . \qquad (8.471)$$

Namely,

$$\sum_{i\in\{i\},f\in\{f\}} |\sigma_{f,i}|^2 = \sum_{i\in\{i\},f\in\{f\}} |\sigma_{\Omega f,\Omega i}|^2 . \qquad (8.472)$$

We are intetrested in transitions that are not possible by $\mathcal{H}^s$ alone; thus,

$$S^s_{f,i} = 0 \quad \rightarrow \quad S_{f,i} = (S^s + \sigma)_{f,i} = \sigma_{f,i} . \qquad (8.473)$$

Then, (8.472) indicates that the transition rate $i \rightarrow f$ is identical to $\Omega i \rightarrow \Omega f$ if the initial and final states are summed over all possible states reachable by strong interaction. This is correct to the first order in σ.

If the initial state is a single particle, then the only state reachable by strong interaction is itself. Then, (8.472) tells us that the decay rate of a particle is the same as that of antiparticle if the decay rates to all final states reachable by strong interaction are summed. For example, the B^+ meson can decay to $K^+\pi^0$, $K^0\pi^+$, $K^{*+}\rho^0$, $K^{*0}\rho^+$ etc. These final states have all stangeness $+1$ and the strong interaction can have nonzero matrix elements among them. The CPT invariance states that the sum of the decay rates should be the same between the B^+ decays to these modes and the corresponding B^- decays to the CPT inverted modes. On the other hand, individual mode may not have the same rate; for example, the rate of $B^+ \rightarrow K^+\pi^0$ mode may not be the same as that of $B^- \rightarrow K^-\pi^0$ which would indicate CP violation. In such cases, CP would be violated while CPT is conserved.

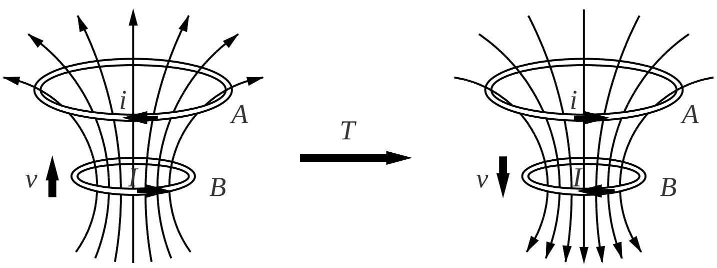
The CPT symmetry is a non-trivial symmetry. In particular, it does not mean that the operator $\Omega = \mathcal{CPT}$ is the identity operator. Yet, it holds with great generality, and its consequences are far-reaching. Since Ω is an antilinear operator, it does not have physical eigenvalues. Thus, it does not lead to conservation of quantum numbers. This is in contrast to the cases of P and C where the quantum numbers of $\mathcal{P}$ and $\mathcal{C}$ are conserved in the course of interactions provided that the interaction is symmetric under the corresponding operations.

When quantum field theory was being constructed, it was thought that the theory is in general symmetric under P, C, and T separately. In 1956, however, it was found that parity was violated in the β-decay of Co^{60}, in the decay $\pi^+ \rightarrow \mu^+\nu$ and also in the muon decay itself. In the decay of Co^{60}, an electron tended to be emitted in the direction of the spin of the nucleus, and the muon in the pion decay was found to have negative helicity. When reflected into a mirror, the electron would be emitted preferentially in the direction opposite to the spin of the nucleus, and the muon would have positive helicity. Thus, these phenomena demonstrated violation of the parity symmetry. In 1964, the combination CP was found to be also violated in the neutral kaon system. The combination CPT symmetry is still considered sacred and so far there is no experimental evidence that it is violated. If history is of any guidance, however, clearly we should be open-minded about CPT symmetry as well.

Problems

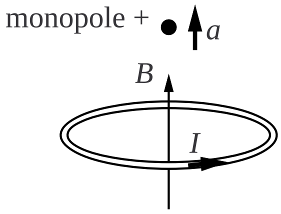
8.1 *Symmetries in physics - definition.*

(*a*) *Lenz's law states that when the flux of magnetic field going through a loop of conductor varies with time, then a current is induced in the loop in the direction such that the flux caused by the induced current compensates the change of the original flux. Let's examine if this law is invariant under time reversal. Assume that there are two coils where coil A is at rest and coil B is moving with a constant velocity as shown in the figure.*



Initially, no current flows in coil A while coil B carries a current. Assume that two coils are heavy enough that the velocities of the coils can be regarded as constant and that the coil B is superconducting such that the current continues to flow without a battery. Then, the upward flux is increasing in the coil A and thus according to Lenz's law a current is induced in the coil A as shown in the figure. The flux created by the induced current is assumed to be small compared to the original flux. Take the motions of coils and electrons that carry the currents and time-reverse them, and ask if they satisfy Lenz's law, then apparently they do not. On the other hand, we know that the electromagnetism does not violate time-reversal symmetry. Explain where we have made a mistake. Note that the symmetry criterion is applied to the motions of electrons and coils only and the magnetic fields are used only to state the law of physics.

(*b*) *The magnetic monopole is a particle that carries the hypothetical charge of magnetic field. The law is that when a monopole of charge g is placed in a magnetic field $\vec{B}$, it feels a force $g\vec{B}$. One such phenomena is shown in the figure below where a heavy and superconducting coil carries a current I and a magnetic monople of positive charge and mass m is placed in the magnetic field which will then accelerate due to* *the force acted on the magnetic charge.*



Apply time reversal to the motions of electrons that carries the current and the magnetic monopole. Is the law of physics invariant under time reversal? Then apply mirror inversion to the original phenomenon. Is the law of physics invariant under mirror inversion? Assume that the sign of the magnetic charge does not change under time reversal or under mirror inversion. Is the law of physics invariant if the magnetic charge changes sign under time reversal or mirror inversion?

8.2 *Construction of the parity operator.*
In the text, we have shown that an operator $\mathcal{P}$ that satisfies $\mathcal{P}a^\dagger_{n\vec{p}\sigma}\mathcal{P}^{-1} = \eta_{n\vec{p}\sigma}\, a^\dagger_{n\,-\vec{p}\,\sigma}$ exists. It is also possible to write $\mathcal{P}$ in a closed form. The action of $\mathcal{P}$ on a single-particle state $\mathcal{P}|n\,\vec{p}\,\sigma\rangle = \eta_n|n\,-\vec{p}\,\sigma\rangle$ indicates that if one kills a particle of $|n\,\vec{p}\,\sigma\rangle$ and creates a particle of $|n\,-\vec{p}\,\sigma\rangle$. So one may try

$$\mathcal{P} = \sum_{n\,\vec{p}\,\sigma} \eta_n a^\dagger_{n\,-\vec{p}\,\sigma} a_{n\,\vec{p}\,\sigma} \quad (?)\,.$$

This works fine for a single particle state, but not for a multi-particle state. We will now show that for a neutral spin-0 particle, and the phase factor $\eta = \pm 1$, the operator $\mathcal{P}$ given by

$$\mathcal{P} = e^{iA}\,, \quad A = -\frac{\pi}{2}\sum_{\vec{p}}(a^\dagger_{\vec{p}}a_{\vec{p}} - \eta a^\dagger_{\vec{p}}a_{-\vec{p}})$$

satisfies $\mathcal{P}a_{\vec{p}}\mathcal{P}^\dagger = \eta a_{-\vec{p}}$.
(a) *Show that the following relation holds,*

$$\underbrace{[A, \cdots [A}_{n\ A's}, a_{\vec{p}}] \cdots] = \frac{\pi^n}{2}(a_{\vec{p}} - \eta a_{-\vec{p}})\,.$$

(b) *Use the identity*

$$e^C B e^{-C} = B + [C,B] + \frac{1}{2!}[C,[C,B]] + \frac{1}{3!}[C,[C,[C,B]]] + \cdots$$

to complete the proof.

8.3 *Existence and uniqueness of the* T *and* Γ *matrices for fermion.*
Suppose γ_μ $(\mu = 0, 1, 2, 3)$ *satisfy the Clifford algebra* $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ *and that* γ^0 *is hermitian and* γ_i *are antihermitian:* $\gamma_\mu^\dagger = \gamma^\mu$. *The Pauli's fundamental theorem states that if there is another set of* 4×4 *matrices* $\gamma^{\mu\prime}$ *that satisfy the Clifford algebra, then there exits a matrix* V *such that* $\gamma^{\mu\prime} = V\gamma^\mu V^{-1}$ *where* V *is unique up to a constant multiplicative factor.*
(a) *Suppose* $\gamma_\mu'^\dagger = \gamma^{\mu\prime}$, *then show that* V *is a unitary matrix, and that it is unique up to a phase factor.*
(b) *Show that a set of 4 matrices* $\gamma_\mu' \equiv -\gamma_\mu^*$ $(\mu = 0, 1, 2, 3)$ *satisfy the same Cilfford algebra and* $\gamma_\mu'^\dagger = \gamma^{\mu\prime}$.
(c) *Use the Pauli's fundamental theorem and its corollary to show that there exists a unitary matrix* Γ *that satisfies* $\Gamma\gamma_\mu^*\Gamma^\dagger = -\gamma_\mu$. *and that it has to be symmetric or antisymmetric:* $\Gamma^T = \pm\Gamma$.
(d) *Repeat the procedure of (b) and (c) to show that there exits a unitary matrix* T *that satisfies* $\mathrm{T}\gamma^\mu\mathrm{T}^\dagger = \gamma_\mu^*$. *which has to be symmetric or antisymmetric:* $\mathrm{T}^T = \pm\mathrm{T}$.

8.4 *Phase-angle relation.*
The general effective interaction Hamiltonian for the neutron beta decay $n \to pe^-\bar{\nu}$ *is given by*

$$\mathcal{H}_{\rm int}(x) = \frac{G_F}{2}\Big(\bar{p}(x)\gamma_\mu(g_V + g_A\gamma_5)n(x)\Big)\Big(\bar{e}(x)\gamma_\mu(1-\gamma_5)\nu(x)\Big)$$

where G_F *is the Fermi coupling constant which is real, and* $p(x)$, $n(x)$, $e(x)$, $\nu(x)$ *are the operator fields for proton, neutron, electron, and neutrino, respectively. The constant coefficients* g_A *and* g_V *are in general complex. Show that if* g_A *and* g_V *are real, then the interaction is invariant under time reversal; namely,* $\mathcal{T}\mathcal{H}_{\rm int}(x)\mathcal{T}^\dagger = \mathcal{H}_{\rm int}(x')$ *where* $x' = (-x^0, \vec{x})$. *Use the tranformation of fermion field* $\mathcal{T}\psi(x)\mathcal{T}^\dagger = \zeta_n\mathrm{T}\psi(x')$. *What it actually means is that if* g_A *and* g_V *are real one can adjust the unphysical phases* ζ_n *so that* $\mathcal{T}\mathcal{H}_{\rm int}(x)\mathcal{T}^\dagger = \mathcal{H}_{\rm int}(x')$ *holds. Experimentally, the phase angle between* g_A *and* g_V *have been measured to be* 180.07 ± 0.18 *degrees.*

8.5 *The quark-W coupling and* T*-violation.*
The quark-W coupling of the standard model is given by

$$\mathcal{H}_{qW} = V_{ij}\,\bar{u}_i\gamma_\mu(1-\gamma_5)d_j\,W^\mu + h.c.$$

where $(u_1, u_2, u_3) \equiv (u, c, t)$, $(d_1, d_2, d_3) \equiv (d, s, b)$, *and* V_{ij} *is a* 3×3 *unitary matrix. Show that this interaction is in general not invariant under time reversal. Namely, one cannot adjust the time reversal phases of the quarks and* W *to make the interaction Hamiltonian satisfy* $\mathcal{T}\mathcal{H}_{qW}(x)\mathcal{T}^\dagger = \mathcal{H}_{qW}(x')$ *with* $x' = (-x^0, \vec{x})$.